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2019S-27

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2019s-27

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Série Scientifique
Scientific Series

Montréal
Novembre/November 2019

CIRANO
Allier savoir et décision

Centre interuniversitaire de recherche en analyse des organisations

ISSN 2292-0838 (en ligne)

How to Manage, Induce, or Prevent Regime Shifts: A Selective Survey

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Abstract/Résumé

How do economic agents manage anticipated shifts in regimes? How do they try to influence or prevent the arrival of such shifts? This paper provides a selective survey of the analysis of regime shifts from an economic viewpoint, with particular emphasis on the use of the techniques of optimal control theory and differential games. We examine the concepts of regime shifts, thresholds, and tipping points in both deterministic and stochastic settings, with or without ambiguity-aversion. Applications to the analysis of political regime shifts are reviewed, with a focus on the role of policy instruments such as repression, redistribution, and gradual democratization. Other dynamic games involving regime shifts that we survey include games of resource exploitation and games in industrial organization theory (R&D races, sabotage against rivals to prevent entry). We compare regime shifts under a Big Push and gradual regime changes.

Keywords/Mots-clés: Regime shifts, Thresholds, Tipping Points, Political Repression, Democratization

JEL Codes/Codes JEL: C61, H11, L13, Q28

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1 Introduction

Many significant changes that occur to human societies, both at the macro level and at the micro level, are often associated with "sudden" shifts in the regimes or the modes of operations.[1] Examples of regime shifts in economics include the introduction of a new technology which makes the old mode of production obsolete (Doraszelski, 2003), changes in the property rights regime (such as the enclosure process which created a landless working class in England), the emancipation of slave labor, revolutions (see, e.g., Campante and Chor, 2012, Lang and De Sterck, 2014, Boucekkine et al., 2016, Michaeli and Spiro, 2019, for models related to the Arab Spring), the transfers of power from a colonial regime to a democratic regime, human-induced climatic changes that can wipe out a large number of species. At the individual level, regime shifts include sudden events which change one's activities and consumption patterns, such as retirement, divorce, serious illness, or conversion to a new faith.

Another source of regime shift is changes in preferences. Kemp and Long (1977) analysed the consequences of a shift in preferences of a decision making body: a peaceful transfer of power anticipated by a colonial administration that plans to hand over the administration of a colony to a democratic government to be elected by local residents. Nkuiya and Costello (2016) argued that a society's environmental preferences may change in the future when the citizens become more acutely aware of costs and benefits of conservation. These preferences changes may themselves be triggered by a series of events. The authors wrote that "The modern environmental movement in the United States, where the Environmental Protection Agency, the Clean Water Act, and the National Environmental Policy Act were all formed over a relatively short period of time, is thought to have been triggered by a series of environmental disasters that raised the environmental profile sufficiently to incite public action" (p. 194). Related works on change in preferences or uncertain future preferences include Le Kama (2001), Beltrati et al. (1998), Le Kama and Schubert (2004), Leonard and Long (2014), and Itaya and Tsoukis (2019).

Regime shifts can occur in the natural environment even in the absence of human activities. For example, lakes may shift from oligotrophic conditions (i.e., exhibiting a deficiency of plant nutrients, such that the water is very clear) to eutrophic conditions (displaying an abundance of nutrients), impacting fish populations and water quality (Scheffer, 1997; Carpenter et al., 1999; Carpenter, 2003, Brock ans Starrett, 2003). Coral reef systems can undergo changes from coral dominated state to algal dominated states. Forested land can become grassland. A biological invasion can wipe out wild and domestic animals and plants (Olson and Roy, 2002). A disease can spread and become persistent after crossing an epidemiological threshold. For analyses of thresholds in epedemic diseases, see Veliov (2005),

[1] From an historical perspective, what is termed "sudden" can correspond to hundred of years. For example, in England, the change in property right regime brought about by the "enclosure" movement took more than 300 years. Between 1605 and 1914, over 5000 "inclosure acts" were passed by Parliament, which transferred to private owners land that was previously common properties. The general question as to whether most changes occur as discrete jumps or in a continuous fashion is a matter of debate, which to some extent hinges on what one means when words such as continuity and suddenness are used. For example, the theory of punctuated equilibrium, put forward by Niles Eldredge and Stephen Jay Gould (1972) as a "better description" of the evolutionary process than Darwin's gradualism has been opposed by Richard Dawkins (1986) on the ground that it was wrong to interpret gradualism as "constant speedism".

Sims et al., (2016), among others.

From an economic view points, regime shifts are often caused by a desire for changes on the part of some powerful coalitions of economic agents in order to further their interests. Throughout human history, many conflicts between nations or between social classes within a nation (e.g., the 'elite' versus the 'citizens') are attributable to attempts of possession or expropriation of natural resources. (See for example Long (1975) on the nationalization of mines; Acemoglu and Robinson (2001, 2006) on class conflicts; van der Ploeg (2010, 2018) on resource wars; and Long (2013) for a review of the theory of contests).[2] Adam Smith (1776) pointed out that the desire to possess more natural resources was one of the motives behind the European conquest of the New World and the establishment of colonies around the globe, some of which thrived on the systematic large-scaled exploitation of slave labor. Many changes that occur in our natural environment (such as climate change, with possible tipping points) can be attributed to the race among industrialised nations to become a dominant actor in the world scene.[3] Conflicts often arise because of lack of well-defined property rights in the exploitation of resources. In fact, the word 'rivals' were derived from the Latin word 'rivales' which designated people who drew water from the same stream (rivus).[4] Indeed, Couttenier and Soubeyran (2014, 2015) found that natural resources played a key role in causing civil conflicts and documented the empirical relationship between water shortage on civil wars in Sub-Saharan Africa.

How do economic agents manage expected shifts in regimes? How do they try to influence or prevent the arrival of such shifts? This chapter provides a selective survey of the analysis of regime shifts from an economic view point, with particular emphasis on the use of the techniques of optimal control theory and differential games.

This paper is organized as follows. Section 2 gives an overview of the concepts of regime shifts, thresholds, and tipping points. Section 3 shows how unknown tipping points affect the optimal current policy of decision makers, with or without ambiguity aversion. Section 4's focus is on political regime shifts in a two-class economy, where we review models of revolution and of how the elite may try to prevent revolution by using policy instruments such as repression, redistribution, and gradual democratization. Section 5 reviews models of dynamic games in resource exploitation involving regime shifts and thresholds. Section 6 reviews some studies of regime shifts in industrial organization theory, with focus on R&D races, including efforts to sabotage rivals in order to prevent entry. Section 7 reviews games of regime shifts when players can manage a Big Push. Section 8 discusses some directions for future research.

[2] The Arab Spring, which undoubtedly has many facets, is not unrelated to the contests for rents between the elite and the citizens.

[3] To be fair, humans are also one of nature's most cooperative species. See for example Seabright (2010), Grafton et al. (2017), and Roemer's book, "How We Cooperate: A Theory of Kantian Optimization", (2019, Yale University Press).

[4] Dictionnaire LE ROBERT, Société du Nouveau Littré, Paris: 1979.

2 Regime Shifts, Thresholds and Tipping Points

In this section we briefly introduce the concepts of regime shift, threshold, and tipping point and give a brief overview of the literature on these topics. More detailed discussions will be provided in later sections.

2.1 Regime shifts

A regime shift is a discrete break in a dynamic system: at the regime-switching time, there is a discrete change in either the objective function, or the transition dynamics. Regime shifts can be anticipated to some extent, and such anticipation affects the behavior of economic agents prior to the actual occurence of the shifts. A prototype model of optimal response to anticipated regime shifts (in the form of an anticipated machine failure) was developed by Kamien and Schwartz (1971). This model predates models of responses to threat of environmental collapses (Cropper, 1976; Reed, 1988). Along the same vein, Long (1975) showed that a monopolist mine owner that expects the nationalization of the mine to occur at some unknown date in the future would hasten his extraction rate. Another early contribution to the modelization of responses to anticipated regime shifts was Kemp and Long (1977). They formulated an optimal control problem with several state variables to show how economic agents would alter their optimal plans if a regime shift is anticipated to take place at some known future date. They mentioned two classes of shifts: a shift in preferences and a shift in technology. In the case of preference shifts, they supposed that a first economic actor (an individual or a group) cares not only about its own present happiness but also about the future happiness of the second individual or groups whose preferences may differ substantially from those of the first actor. As an example of preference shift, they considered the task of an imperial power committed to the eventual independence of its colony. Another example of optimal actions under an anticipated technology shift is that a firm knows that at some future point in time a patent will expire and a new process becomes available to it.

In Kemp and Long (1977), at each regime-shift time t_i, there are H inequality constraints involving the n state variables:

$$S_h^i(x_1(t_i), ..., x_n(t_i)) \geq 0,\ h = 1, 2, ..., H.$$

They proved that the co-state variables are continuous at the time of the shift, unless one of the constraints (say $S_{h^*}^i$) is binding, in which case the co-state associated with the state variable x_j will jump downwards if and only if $\partial S_{h^*}^i / \partial x_j(t_i) \geq 0$. In the case of a parent who expects to transfer the family business to an offspring, Kemp and Long (1977) showed that prior to the shift time, the consumption path may be non-monotone. They also considered the case of a mining firm that expects a discrete shift in the price. Anticipation of the price shift makes the firm modify its current extraction plan. This result has implications to what is now known as the Green Paradox: announcing policies that are intended to mitigate climate changes (e.g. policies that institute future sharp rises in carbon tax) may worsen the climate, as the anticipation of the future taxe hike induces coal and oil producers to quickly exhaust of their stocks of fossil fuels, resulting in the unintended consequence of hastening the rise in global temperature (Long and Sinn, 1985; Sinn 2012).

A related work on regime shifts is Hillman and Long (1985),who considered an economy that owns a small stock of exhaustible resource (say oil) which competes in the domestic market with imported perfect substitutes. The shift in question is change from the free-trade regime to an embargo regime, in which the economy is subject to a trade embargo by, say, a foreign oil cartel. Unlike Kemp and Long (1977) who assumed a known date of regime shift, Hillman and Long (1985) supposed that the date when a foreign embargo will be imposed on the home country is a stochastic variable. They proved that the planner of the economy that anticipates the threat of embargo will extract its oil more conservatively. Interestingly, if the economy's resource stock is exploited by perfectly competitive domestic firms, these firms will replicate the planner's conservationist solution, because they anticipate an upward jump in domestic oil price as soon as the embargo occurs. A striking result is that if the domestic stock is exploited by a single firm, then the embargo threat will cause this firm to overextract the resource in the pre-embargo phase, in anticipation that it will become a monopolist in the domestic market immediately after the embargo takes place. This over-extraction prior to the embargo can be explained intuitively: the monopolist's profit under the embargo regime is higher, the lower is its stock at the time of the regime shift. Interestingly, in this scenario, Robert Solow's dictum (that the monopolist is the conservationist's best friend) fails to hold.

In both Kemp and Long (1977) and Hillman and Long (1985), it was assumed that the date of regime shift, whether known or stochastic, is outside the control of the planner. The case of where a regime shift date can be chosen by the planner was considered by Hung et al. (1984) using a model of transition from fossil energy to an non-exhaustible substitute (such as solar energy). The date of transition, t_T, is optimally chosen to maximize the discounted stream of social welfare, by balancing the cost of investment with the discounted future stream of benefits derived from consuming the clean substitute. At the date of transition, a lumpy cost K must be incurred. The authors show that the transition date is determined by the condition that the current-value Hamiltonian immediately before the transition, $H(t_T^-)$, is smaller than that the Hamiltonian immediately after the transition, $H(t_T^+)$, by the amount rK, where r is the interest rate. This implies that the equilibrium price path of energy has a discrete downward jump at the time of regime shift. As expected, the regime-shift decision involves a trade-off, since adopting a new regime brings immediate costs as well as future benefits. The result of the paper by Hung et al. (1984) is consistent with the multi-stage optimization analysis of regime switching by Tomiyama (1985) and Amit (1986) which also endogenously determine switching times.

The analysis of dynamic responses to regime shifts can be conducted using two approaches: The optimal control theory/ dynamic programming approach (where a single decision maker decides how to cope with an anticipated regime shift), and the dynamic games approach, where multiple agents plan their responses in a non-cooperative way, while strategically reacting to one another. The first approach is clearly simpler, but it misses out some important strategic considerations. Representative papers using the first approach include Tsur and Zemel (1996, 1998), Ren and Polasky (2014), Nkuiya and Costello (2016), and Lemoine and Traeger (2016), among others. Papers using the dynamic games approach include Tornell (1997), Mäler et al. (2003), Doraszelski (2003), Dawid et al. (2015), and Long et al. (2017). A paper that does not include differential games but does take into account game-theoretic considerations is Nakuiya et al. (2015), where the threat of regime shift occurs only in the first period. They found that countries are more likely to ratify a climate change today when they face endogenous uncertainty about a possible future upward shift in damage costs.

2.2 Thresholds and Tipping Points

Quite often, a regime shift occurs when a certain endogenous variable crosses some thresholds, the exact value of which may, or may not, be known to a decision maker. One may make a distinction between two types of thresholds in the state variable of a dynamic system. The first type of threshold, once crossed, induces a *discrete* change in the differential equation (or difference equation) that describes the state dynamics, or in the preferences of the decision makers. The second type of threshold (as in the case of the shallow lake problem that will be discussed in subsection 5.2 below) only involves, upon crossing it, a change in the basin of attraction, or a substantial change in the qualitative properties of the optimal policy. Thresholds play an important role in economic models of social change. The proverbial "last straw that breaks the back of the donkey" is a case in point. In an interesting article on racial segregation, Schelling (1971) showed that a small change in the initial mixture of blacks and whites in a neighbourhood may eventually lead to a complete segregation. If there is a limit to how small a minority the members of either color are willing to be, for example, a 25% minority, then "initial mixtures ranging from 25% to 75% will survive but initial mixtures more extreme than that will lose their minority members and become all of one color" (p. 148). His models contributed to the explanation of the phenomenon called " neighbourhood tipping", which occurs "when a recognizable new minority enters a neighborhood in sufficient numbers to cause the earlier residents to begin evacuating" (p. 181).

Another early interesting work on threshold is that of Azariadis and Drazen (1990). They show that the success or failure of a developing economy depends on whether it manages to pass a certain threshold level of externalities. Similarly, in the context of the tragedy of the commons, Lasserre and Soubeyran (2003) found that a small amelioration of institutions can move an economy to a superior equilibrium. Along the same vein, Leonard and Long (2012) demonstrated how a strengthening of the enforcement of property rights, financed by taxation supported by a self-interested electorate, could move the economy to an efficient steady state. These papers assume that economic agents care only about their material wellbeing. As a counterpoint, Long (2019) offers a model where economic agents care also about their self-image. Long assumes that economic agents feel bad if their action falls short of the Kantian ideal. Using an overlapping generations model in which pro-social attitudes evolve across generations Long (2019) shows that there is a threshold level of pro-socialness beyond which the economy will converge to a steady state with a high level of both prosocialness and material prosperity, while below the threshold, society's level of pro-socialness will eventually vanish, and the economy will end up in poverty.

Quite often while the decision maker is aware of the possibility of thresholds and tipping points, there is considerable uncertainty as to the exact location of the tipping points. This is a particularly relevant issue in the analysis of optimal responses to climate change. Heal (1984) and Tsur and Zemel (1996) assume that the decision maker has imperfect knowledge of the underlying climate threshold. Keller et al. (2004) study optimal economic growth under uncertain climate thresholds. While Keller et al. (2004), Gjerde et al. (1999), and Lontzek et al. (2015) model climate tipping points as directly reducing output or utility, Lemoine and Traeger (2014, 2016) and van der Ploeg (2014) model tipping points as a shift in the dynamics of the climate system.

The use of optimal control theory enriches the analysis of thresholds. Skiba (1978) showed that if an optimal control problem exhibits two steady states that are locally stable in the saddlepoint sense, then there exists in the state space a threshold that separates the two basins of attraction. Later authors call such a threshold a "Skiba point". There is a large literature on Skiba points (see, e.g., Feichtinger and Wirl, 2000; Wagener, 2003; Hartl et al., 2004; Wirl and Feichtinger, 2005; Wirl, 2016; Yanase and Long, 2019).

A key feature of a Skiba point is that at such a point, the decision maker is indifferent between two trajectories, each converging to a different steady state. For example, Hartl and Kort (1996) show that a firm facing an emission tax may choose between achieving a steady state with a high capital stock which is compatible with efficient abatement efforts, or a low capital stock with no abatement efforts. The firm has to invest more to reach the high capital stock equilibrium. This implies that there is a discontinuity of the policy function at the Skiba point k^S. Immediately to the right of k^S, the firm invests a great deal more than to the left of k^S. Discontinuity, however, is not a generic feature of Skiba point. Wirl and Feichtinger (2005) show the existence of a Skiba point with a continuous policy function. This requires that the unstable steady state is a node (rather than a focus). Hartl et al. (2004) give a complete classification of Skiba points near unstable steady states: focus, continuous node, and discontinuous node. These papers assume that there is a single decision maker. When there are several decision makers interacting in a dynamic game, the study of Skiba points becomes much more complicated. See Section 6 for details.

3 Unknown Tipping Points: The hazard rate function approach

The precise points at which tipping may occur are typically unknown, because of lack of scientific information (Lemoine and Traeger, 2014). A standard approach to model unknown tipping points is to use the hazard function approach (Clarke and Reed, 1994; Tsur and Zemel, 1998; Gjerde et al., 1999; de Zeeuw and Zemel, 2012). For example, in the context of risks of abrupt climate change that are associated with the stock of green house gases (GHG), one could imagine that an adverse climatic event may occur at some unknown time T in the future that would inflict severe economic damages. A convenient formulation is to suppose that the distribution of the random occurrence date T is related to a hazard rate function $h(X)$ where $X(s) \geq 0$ is the stock of GHG at date s, with $h(X) > 0$ for $X > 0$. Given that the adverse climatic event has not occurred at or before time t_0, for any given $t > t_0$ the probability that T will occur after time t is specified as follows:

$$\Pr\left(T > t \middle| \, T > t_0\right) = e^{-\int_{t_0}^{t} h(X(s))ds}.$$

The conditional probability that the adverse event occurs before some time $t > t_0$ is

$$F\left(t \middle| \, t_0\right) = 1 - e^{-\int_{t_0}^{t} h(X(s))ds}$$

Notice that this formulation implies that, assuming that $h(X)$ is strictly positive, the event is definitely going to take place at some time in the future:

$$\lim_{t\to\infty} F\left(t|\ t_0\right) = 1.$$

The corresponding conditional density function is

$$f\left(t|\ t_0\right) = F'(t|\ t_0) = h(X(t))e^{-\int_{t_0}^{t} h(X(s))ds}.$$

Thus, at time t_0, the conditional probability that the adverse event will occur at some time during the time interval $(t_0, t_0 + \Delta t)$ is approximately

$$h(X(t_0)) \times \Delta t$$

provided that Δt is sufficiently small.

In general, the event needs not be a climatic event, and the state variable X needs not refer to the stock of GHG. Thus, X could refer to, say, the stock of fish in a fishing ground, and the event could be a collapse of the fish stock, or a change in its growth function, $G(X)$. Or X could simply be time, as in the nationalization model of Long (1975). While it is usually assumed that the hazard rate depends only on the state variable, some authors have allowed the hazard rate to depend on both a state variable and a control variable, under the assumption that the feedback control rule is continuous in the state variable. See for example Doraszelski (2003, p. 22), van der Ploeg (2018) and Haurie et al. (2012).

The hazard rate approach can be applied to a single occurence or to recurrent events. See Tsur and Zemel (1998) for the distinction. For an analysis of recurrent environmental catastrophes, see Tsur and Zemel (2016), where increased GHG concentration implies higher frequency of occurrence. They focus on long run properties, using techniques developed in Tsur and Zemel (2017).

3.1 The ambiguous effect of anticipation of regime shifts

How does the possibility of a regime shift influence the behavior of the decision maker? In general, the answer to this question is ambiguous. We can illustrate this ambiguity by considering a model of a fishery where the regime shift takes the form of a change in the natural growth rate of the stock, from $G_1(.)$ to $G_2(.)$, where $G_2(X) < G_1(X)$ for all $X \geq 0$. The special case where $G_2(X)$ is identically zero corresponds to a stock collapse (i.e., the fish stock X suddenly becomes zero at a random date T). Let us consider the fishery model of Polasky et al. (2011), where an analytical solution can be obtained thanks to the authors' assumption that the instantaneous payoff function is linear in the harvesting rate, y. Before the regime shift, taking into account human's exploitation of the fish stock, the net rate of growth of the fish stock is

$$\dot{X} = G_1(X) - y.$$

Let us first consider the optimal harvest policy if the decision maker is certain that there will never be a regime change. Let $r > 0$ be the rate of discount. The decision maker's objective is to maximize

$$\int_0^\infty e^{-rt} py(t)dt$$

where $p > 0$ is a constant. Assume that $G(.)$ is hump-shaped, with $G(0) = 0$, $G'(0) > r$, $G'' < 0$, and $G'\left(\overline{X}\right) = 0$ for some $\overline{X} > 0$. Assume that $0 \leq y \leq y_m$ where y_m is an exogenous upper bound on y, with $y_m > G\left(\overline{X}\right)$. As an example, one can assume that

$$G_i(X) = X\left(1 - \frac{X}{K_i}\right)$$

where $K_i > 0$ is called the carrying capacity. Under these assumptions, it is well-known that, with no threat of regime shift, the decision maker will aim at a steady state stock X^* such that $G'(X^*) = r$, and that the optimal y is equal to zero for $X < X^*$, and is equal to y_m for $X > X^*$. (See Clark, 1990).

What happens if there is a threat of a regime shift from $G_1(.)$ to $G_2(.)$ as specified above? Assume that the hazard rate function is $h(X) \geq 0$, with $h'(X) \leq 0$ (i.e., the risk is lower when the stock is larger).[5] What would be the steady state stock that the decision maker aims at? Let us call this stock X_1. Is X_1 greater than or smaller than X^*? Polasky et al. (2011) show that the answer is ambiguous if $G_2(X) \equiv 0$, unless additional assumptions are made. To understand this ambiguity, recall that the HJB equation when the system is in regime 1 is given by

$$rV_1(X) = \max_{0 \leq y \leq y_m} \left[py + V_1'(X)(G_1(X) - y)\right] + h(X)\left[V_2(X) - V_1(X)\right] \tag{1}$$

where $V_i(.)$ is the value function under the i^{th}regime.[6] (See e.g., Dockner et al. (2000) for a general formulation of this type of regime shifts).

If the system is in regime 1, when one maximizes the right-hand side of the HJB equation with respect to y, the optimal harvesting effort is $y = 0$ for all X such that $V_1'(X) > p$ and $y = y_m$ if $V_1'(X) < p$ (with y indeterminate if $V'(X) = p$). One searches for a value $X_1 < K_1$ such that $y = 0$ for $X < X_1$ and $y = y_m$ for $X > X_1$. Then eq. (1) yields

$$0 = V_1'(X)G_1(X) + h(X)V_2'(X) - \left[r + h(X)\right]V_1(X) \text{ for } X < X_1 \tag{2}$$

$$0 = py_m + V_1'(X)(G_1(X) - y_m) + h(X)V_2'(X) - \left[r + h(X)\right]V_1(X) \text{ for } X > X_1 \tag{3}$$

Assuming that $V_1'(X)$ is continuous at X_1, the two eqs. (2) and (3) yield

$$V_1'(X_1) = p \text{ and } V_1(X_1) = \frac{pG(X_1) + h(X_1)V_2(X_1)}{r + h(X_1)} \tag{4}$$

Furthermore, assume that $V_1''(X)$ exists. Then differentiating eqs. (2) and (3) with respect X, one obtains

$$G_1(X)V_1''(X) = \phi(X) \text{ if } X < X_1$$

and

$$\left[G_1(X) - y_m\right]V_1''(X) = \phi(X) \text{ if } X > X_1$$

where

$$\phi(X) \equiv \left[r + h(X) - G_1'(X)\right]V_1'(X) - h(X)V_2'(X) + h'(X)\left[V_1(X) - V_2(X)\right]$$

[5] Polasky et al. (2011, p. 233) assume that $h'(X) = 0$ at $X = K_1$.

[6] Note that in writing the above HJB equation, it is assumed that $V_1'(X)$ is defined for all $X > 0$.

Under the assumption that $V_1''(X_1) \leq 0$, one can see that $\phi(X)$ is negative to the left of X_1 and positive to the right. The assumed continuity of V_1' and V_2' then implies that $\phi(X_1) = 0$. This equation and (4) taken together imply that

$$G_1'(X_1) = r + h(X_1)\left[1 - \frac{V_2'(X_1)}{p}\right] + \frac{h'(X_1)}{r + h(X_1)}\left[G_1(X_1) - \frac{r}{p}V_2(X_1)\right] \tag{5}$$

Eq. (5) shows that in general one cannot determine whether the (regime 1) steady state stock X_1 exceeds or falls short of the steady state stock X^* (which applies if there is no threat of regime shift). To see this ambiguity, consider the case where $G_2(X) = 0$ identically (i.e., the stock collapses immediately after the adverse event occurs), so that $V_2(X) = 0$ identiccaly. Consider two benchmark subcases. First, the subcase where $h(X) = \lambda$, a positive constant. Then eq. (5) gives $G_1'(X_1) = r + \lambda > r = G_1'(X^*)$. That is, under the threat of an exogenous regime shift, the planner's exploitation is less conservationist than under the no-threat scenario. (This is reminiscent of the result of Long (1975): the threat of nationalization leads to more aggressive extraction of the mine.) Second, the subcase where $h'(X) < 0$. Then eq. (5) gives

$$G_1'(X_1) = r + h(X_1) + \left\{\frac{h'(X_1)G_1(X_1)}{r + h(X_1)}\right\}$$

Since the term inside the curly bracket is negative for $X_1 < \overline{X}$, we can no longer be sure that $G_1'(X_1) > r$. Thus it is possible that $X_1 > X^*$, i.e., the decision maker's exploitation is more conservationist, because she wants to achieve a lower hazard rate at the steady state of regime 1.

The above "ambiguity result" is in line with previous works for the cases of threats of forest fire and fishery collapse (Reed, 1987, 1988), nuclear power risks (Aronsson et al., 1998; Mähler and Li, 2010), and environmental threats (Clarke and Reed, 1994; Tsur and Zemel, 2006, 2008).

3.2 Knightian Uncertainty: Decision making under ambiguity about tipping points

In many real world problems, such as climate change, our knowledge is so thin that it may not be appropriate to use models that assume a known distribution of stochastic shocks. The terms "Knightian uncertainty" or "deep uncertainty" and "ambiguity" have been used interchangeably to refer to situations in which the underlying probabilities are not known. A number of studies have explored the implications of Knightian uncertainty in the context of climate change. Lange and Treich (2008) use a two-period model to show that ambiguity aversion about damages induces the decision maker to opt for lower emissions. A number of authors use aversion to Knightian uncertainty to motivate the robust control approach to abatement policies (Li et al., 2014, Anderson et al. 2014).

Lemoine and Traeger (2016) analyse the effect of ambiguity aversion on optimal policy in the face of an unknown tipping point. Their point of departure is a model of rational behavior under deep uncertainty that was axiomatized in Traeger (2010), which is closely related to the recursive smooth ambiguity model of Klibanoff et al. (2005, 2009). In Lemoine and Traeger (2016), the vector of state variables is denoted by S_t. This vector can include the capital stock, temperature level, carbon dioxide, and time. The vector of control variables is denoted by x_t. In each period, there is a deterministic utility flow $u_t = u(x_t, S_t)$. The decision maker maximizes the expected intertemporal payoff over the infinite horizon. There are two value functions, $V_0(S)$ and $V_1(S)$, which apply to the pre-tipping world and the post-tipping world. The system dynamics are described by $S_{t+1} = G_0(x_t, S_t)$ for the pre-tipping world and $S_{t+1} = G_1(x_t, S_t)$ for the post-tipping world.[7] In the absence of ambiguity, $V_0(S)$ is related to $V_1(S)$ through the Bellman equation:

$$V_0(S_t) = \max_{x_t} \{u(x_t, S_t) + \beta [(1 - h(S_t, S_{t+1}))V_0(S_{t+1}) + h(S_t,S_{t+1})V_1(S_{t+1})]\} \tag{6}$$

subject to

$$S_{t+1} = G_0(x_t,S_t)$$

where $\beta \in (0, 1)$ is the discount factor, and $h(S_t, S_{t+1})$ is the hazard rate function which gives the probability of that tipping occurs in period $t + 1$.

In the context of climate change, Lemoine and Traeger (2016) define ambiguity as the decision maker's lack of confidence in the hazard rate function $h(S_t, S_{t+1})$. They propose to capture this lack of confidence by introducing into the recursive utility model a concave function, which I denote by $\Phi(.)$, such that the Bellman equation is modified as follows:

$$V_0(S_t) = \max_{x_t} \left\{u(x_t, S_t) + \beta\Phi^{-1} \left[(1 - h(S_t, S_{t+1}))\Phi(V_0(S_{t+1})) + h(S_t,S_{t+1})\Phi(V_1(S_{t+1}))\right]\right\} \tag{7}$$

Since the model cannot be solved analytically, Lemoine and Traeger resort to numerical simulations. To facilitate the simulations, they assume that the function Φ contains two parameters, $\eta \geq 0$ and $\gamma \geq 0$, where η is a measure of aversion to risk and γ takes into account the decision maker's aversion to ambiguity.

They consider two different classses of models. In the first class of models, when a tipping point is crossed, there is a sudden increase in the climate feedbacks that amplify global warming. This type of tipping points increases the effects of emissions on temperature.[8] In the second class of models, a tipping point triggers an increase in the decay rate of CO_2, i.e., a weakening of the carbon sinks.[9] Numerical simulations show that in either class of model, an increase in ambiguity aversion (an increase in γ) will increase the optimal carbon tax and reduce the peak level of CO_2 along an optimal path. The authors decompose the total effect of an increase in γ into two effects: (a) the marginal hazard rate effect (MHE), which reflects the awareness that present policies influence the chance of tipping, and (b) the differential welfare impact (DWI), which compares the effects of abatement on pre-tipping welfare and on post-tipping welfare. The sign of DWI is ambiguous. In the numerical

[7] In their formulation, they also add a random variable ε_t that represents stochastic shocks with a known distribution (these shocks are not ambiguous). For simplicity of exposition, I have omitted this variable.

[8] In the standard DICE model of Nordhaus (2008) without tipping points, there is a parameter called "climate sensitivity", defined as the equilibrium warming from doubling the stock of GHGs. Lemoine and Traeger (2016) introduce Knightian uncertainty about a climate-feedback tipping point that increases this parameter from its pre-tipping value of 3 degrees C to, say, 5 degrees C.

[9] The authors assume that when the unknown tipping point is crossed, there is a sudden decrease in the rate of transfer of CO_2 out of the atmosphere.

calculations, aversion to Knightian uncertainty increases the contribution of MHE to the carbon tax, but tends to reduce the carbon tax via the DWI effect. However, the overall effect of aversion to Knightian uncertainty is to increase the carbon tax.

4 Preventing Regime Shifts: The role of repression, redistribution, and education in a two-class economy

There is a large literature on the threat of revolution that an autocratic regime faces. The early theoretical models of revolutions (Granovetter, 1978; Kuran, 1989) abstract from strategic considerations.[10] More recent works, such as Acemoglu and Robinson (2000, 2001, 2006), offer models on interaction between the ruling elite and the citizens, where coups and revolutions can occur in response to exogenous economic shocks. In Acemoglu and Robinson (2001), there are two groups of agents, the poor and the elite. Each group consists of infinitely-lived individuals. The elite has more capital than the poor. The majority of people are poor, and initially it is the elite that has the political power. The poor can attempt a revolution at any time, but revolution is costly (a fraction of national income is destroyed). If a revolution is successful, a fraction of assets of the elite is expropriated. The elite can avoid a revolution by embarking on a process of democratization. The productivity of capital is a random variable, which is revealed at the beginning of each period: it can be low or high. This random variable affects the opportunity costs of revolution in a nondemocracy as well as the elite's opportunity costs of mounting a coup to overthrow a democracy. These models typically assume that the elite are killed or evicted during or after a successful revolution.

In contrast, Boucekkine et al. (2016) assume that after the elite are removed from power, they co-exist with the citizens and share access to the country's stock of resources. Boucekkine et al. (2016, p. 189) argue that this is consistent with what happened in countries such as Tunisia, where "the Arab Spring events have successfully overthrown the ruling dynasty but have failed to renew the political and economic life to a large extent." Their paper models the efforts of the elite to prolong their regime as much as possible. The elite has two policy instruments: repression and redistribution.[11] The model is solved in two stages. In the post-revolution stage, the elite and the citizens have equal access to the country's resources, and they solve a differential game of resource exploitation in the manner postulated by Tornell and Lane (1999). In the pre-revolution stage, the authors assume a Stackelberg model of differential game. In this game, the elite (the Stackelberg leader) is able to commit to a redistribution parameter, $1 - u_E$, and a represssion parameter r_E, while the citizens, taking these parameters as given, choose the date T at which they start a revolution.[12] Revolution is costly: it destroys a fixed amount, χ, of the country's capital stock, and on top of that, the citizens must incur a direct switching cost (DSC), ψ. This

[10] The non-strategic approach is also taken in a recent interesting paper by Michaeli and Spiro (2019), where they show a number of interesting results, including a demonstration of how the implementation of popular policies, such as Perestroika, can trigger a revolution.

[11] In the model of Boucekkline et al. (2016), these are parameters that the elite chooses at the beginning of the program; they are not control variables in the standard sense of being piece-wise continuous functions of time.

[12] The fraction of national income that is distributed to the citizens is $1 - u_E$.

cost is an increasing and concave function of the level of repression, r_E. It is found that the date T is increasing in $1 - u_E$ and in χ, and decreassing in the economy's initial resource stock. Knowing how the citizens' choice of revolution time depends on the redistribution parameter and the repression parameter, the elite sets these parameters to maximize their own payoffs. This is a deterministic optimal control problem. The authors show that if the vulnerability of the economy is high, the revolution will occur in finite time. However, if the vulnerability is intermediate, in equilibrium the dictatorship survives.

The model by Boucekkine et al. (2016) allows the ruling class to resort only to two policy instruments: repression and redistribution. In a follow-up paper, Boucekkine et al. (2019) consider a third policy instrument that can help the elite prevent a violent revolution: education of the mass that eventually leads to a peaceful handover of power. They develop a dynamic optimization model that portrays the ruling class's policy choice to cope with the threat of revolution. In this model, the elite may choose between (a) keeping the population largely uneducated, while redistributing income just enough to avert a revolution, and (b) embarking on a path of education and development and eventually relinquishing autocratic power, ensuring a smooth democratic transition. In their model, from the point of view of the elite ruling class, education of the oppressed class has two opposing effects. On the one hand, the life-satisfaction threshold above which the population would not revolt is increasing in education, i.e., a more educated mass tends to demand higher life-prospects at the expense of the elite. On the other hand, education contributes to economic development and is conducive to a political culture of negotiation and a recognition of the merit of trying to achieve a compromise (Lipset, 1959, 1960; Barro, 1999; Bourguignon and Verdier, 2000).

In Boucekkine et al. (2019), the ruling class derives income from natural resources (available in a fixed quantity, R, per period). The working class's income consists of wage income, wH, and a transfer Θ from the government. Here, H is the level of human capital, which accumulates as a result of education, E, that the ruling class provides. If the sum $wH + \Theta$ is below a certain threshold, a revolution will take place. This threshold is increasing in the level of human capital. By choosing Θ and by influencing H through education expenditure, the elite can avoid a revolution and stay in power for ever. However, it may be to the elite's advantage to relinquish power at some planned date T through a process of democratization, if the anticipated payoff to the elite at the time of handover, $S(H(T))$, is sufficiently attractive. Boucekkine et al. (2019) assume that this payoff is increasing in $H(T)$. The elite class chooses consumption, $C(t)$, redistribution, $\Theta(t)$, education expenditure, $E(t)$, and a terminal time, T, to maximize its intertemporal welfare, subject to the no-revolt constraint. Their intertemporal welfare is

$$U = \int_0^T e^{-\rho t} u(C(t))dt + e^{-\rho T} S(H(T))$$

This is a standard optimal control problem. The authors show that, depending on parameter values, the optimal solution may be one of three possible varieties: (i) permanent dictatorship with no education, regardless of the initial H_0, (ii) education-driven democratization, with a finite time for power hand-over, also regardless of the initial H_0, and (iii) human-capital poverty trap: there exists a threshold level $\overline{H}$ such that if $H_0 < \overline{H}$ then permanent dictatorship is optimal for the elite, and if $H_0 > \overline{H}$ then democratization through education is optimal.

5 Dynamic Games involving Natural Resources with Threat of Regime Shifts and Thresholds

In this section, we review some dynamic game models of natural resource exploitation that feature either a threat of regime shift, or a threshold.

5.1 Extraction of an exhaustible resource under threat of regime shift

Laurent-Lucchetti and Santugini (2012) study a dynamic game model of common property exhaustible resources under uncertainty about full or partial expropriation, generalizing the nationalization model of Long (1975). Consider a host country that allows two firms to exploit a common resource stock under a contract that requires each firm to pay the host country a fraction τ of its profit. Under the initial agreement, $\tau = \tau_L$. However, there is uncertainty about how long the agreement will last. The host country can legislate a change in τ to a higher value, τ_H. It can also evict one of the firm. The probability that these changes occur is exogenous. Formulating the problem as a dynamic game between the two firms, in which the risk of expropriation is exogenous and the identity of the firm to be expropriated is unknown *ex ante*, the authors find that weak property rights have an ambiguous effect on present extraction. Their theoretical finding is consistent with the empirical evidence provided by in Bohn and Deacon (2000).

How does the threat of being removed from office influence a government's extraction path of an exhaustible resource stock and its exploration efforts? A recent paper by van der Ploeg (2018) offers three related models that shed light on this question. In Model 1, an incumbent faces the threat of removal from office (for ever) by a rival faction. This model is related to the model of the effects of political uncertainty about nationalization (Long, 1975; Konrad et al., 1994; Bohn and Deacon, 2000; Laurent-Luchetti and Santaguni, 2012). The author assumes that the incumbent government (player A) faces the risk of being overthrown by a rival faction (player B). The hazard rate is a constant, $h > 0$. Once player A is removed from office, it receives a smaller share of the resource rent. Under this scenario, it is found that resource extraction by the incumbent is more voracious. Furthermore, the incumbent tends to invest less in the exploration for the resources, because of the hold-up problem.

In Model 2, van der Ploeg (2018) considers the scenario of on-going political resource conflict cycles betweern two political factions. Once a faction is in office, it faces a harzard rate h of being removed by the other faction. After being removed, the faction can regain office, also with the hazard rate h. The author assumes that both factions are obbliged to share equally the resource rents, but the faction that is in office enjoys utility more. This is captured by introducing a multiplicative partisan in-office bias, $\beta > 1$, in line with Aguiar and Amador (2011). The author shows that with perennial on- going political cycles, resource depletion is rapacious especially if the partisan in-office bias is large (high β) and there are frequent changes of government (high h).

In Model 3, the author endogenizes the hazard rate. Again, there are two factions, A and B. If faction A is the incumbent, it faces the hazard rate h^A of being removed from office. Being in office, it can choose the resource extraction rate R^A, obtains the resource rents $\pi(R^A)$, of which a fraction $\tau < 0.5$ must be transferred to the other faction (according to some constitutional convention).

Assume that h^A is a function of A's defence effort, f^A, and of B's attack effort, f^{B*}. Using the common formulation of the rent-seeking literature, assume that

$$h^A = H \frac{(f^{B^*})^\phi}{(f^A)^\phi + (f^{B^*})^\phi} \text{ and } h^B = H \frac{(f^{A^*})^\phi}{(f^B)^\phi + (f^{A^*})^\phi}$$

where H is a constant, and $\phi \in (0, 1)$. Each faction has a maximum of N units of efforts, and the income derived from $N - f$ is $w(N - f)$, where $w > 0$ is the wage rate. Let S denote the stock of the exhaustible resource. Let $V^A(S)$ and $V^{A*}(S)$ denote respectively faction A's value function when it is the incumbent and when it is not the incumbent. Then the HJB equations for A are:

$$rV^A(S) = \max_{f^A, R^A} \left\{ \beta(1-\tau)\pi(R^A) + w(N - f^A) - V_S^A(S)R^A + h^A \left[V^{A^*}(S) - V^A(S) \right] \right\}$$

$$rV^{A^*}(S) = \max_{f^{A*}} \left\{ \tau\pi(R^B) + w(N - f^{A^*}) - V_S^{A*}(S)R^B - h^B \left[V^{A*}(S) - V^A(S) \right] \right\}$$

Faction B is in a similar situation. Assuming that the function $\pi(R)$ is iso-elastic, the value functions can be solved analytically. It is found that dynamic resource wars are more intense if S is high and w is low. Depletion of the reserves is less rapid if τ is closer to 0.5, and if the government's stability is high (a low H). An increase in the partisan in-office bias paramer β leads to more rapacious extraction.

5.2 Dynamic games involving natural resources with thresholds and non-linear dynamics

Examples of dynmaic games involving natural resource stocks with non-linear dynamics include fishery games, and lake-pollution games. Most fishery models assume that the transition equation is concave in the state variable. Even so, multiple steady-state equilibria can exist in concave optimal control fishery problems (see Long, 1977, where it is found that there are three steady-state equilibria, of which the middle one is unstable). Limit cycles can also be optimal (Long, 1992a, 1992b, pp. 294-295; Kemp et al. 1993). The lake-pollution game model is another interesting example of multiple equilibria, where the transition equation is neither concave nor convex in the state variable. This implies that there are potentially several steady states. We describe below a lake-pollution model based on Mäler et al. (2003).

The state variable, $s(t)$, denotes the amount of phosphorus sequestered in algae. There are n players. Player i discharges $c_i(t) \geq 0$ units of phosphorus to the lake. The transition equation is

$$\frac{ds}{dt} = -\delta s(t) + \left[\frac{s^2(t)}{s^2(t) + 1} \right] + \sum_{i=1}^{n} c_i(t), \qquad x(0) = x_0 \geq 0.$$

where $\delta > 0$ and $s(t) \geq 0$. The term inside the square brackets is the internal release of phosphorus that has been sequested in sediments and in submerged vegetation; this term is bounded above by 1. Thus, for any given constant aggregate discharge $C \equiv \sum_{i=1}^{n} c_i$, the steady-state stock of pollution is bounded above by $(C+1)/\delta$. The transition equation can be re-arranged to yield

$$(s^2+1)\frac{ds}{dt} = -\delta s^3 + (1+C)s^2 - \delta s + C \equiv h(s;C,\delta)$$

Since $h(0;C,\delta) = C > 0$ and $h(\infty;C,\delta) = -\infty$, there exists at least one positive steady state.

Suppose c_i is constant. Then it can be shown that, provided $0 < \delta < 3\sqrt{0.375}$, there exists a certain range of c_i such that there are three steady states, denoted by s_L, s_M and s_H where $s_L < s_M < s_H$, where s_M is unstable, and s_L and s_H are locally stable. (In the lake-pollution literature, s_L is usually refered to as the oligotrophic state, and s_H is the eutrophic state.)

Suppose initially the system is at the low steady state s_L. Coinsider a temporarily sustained increase in c_i. If this increase crosses a threshold level, there will be a sudden flip to s_H. This is called a tipping point. If $\delta \le 1/2$, the flip is irreversible, since c_i cannot be negative.[13] In what follows, we assume $1/2 < \delta < 3\sqrt{0.375}$.

Suppose that the net benefit function of player i is

$$B_i = \ln c_i - \omega s^2$$

where $\omega > 0$. Player i's overall payoff is

$$\int_0^\infty e^{-\rho t}\left[\ln c_i - \omega s^2\right] dt$$

Let us compare the open-loop Nash equilibrium with the Markov-perfect Nash equilibrium of this game.

Under open-loop behavior, the Hamiltonian for player i is

$$H_i = \ln c_i - \omega s^2 + \psi_i\left[\frac{s^2(t)}{s^2(t)+1} - \delta s + c_i + (n-1)c_j\right]$$

Assuming a symmetric Nash equilibrium, so that $c_i = c_j = c$, and defining $C = nc$, the necessary conditions are

$$\frac{1}{c_i} + \psi_i = 0$$

$$\dot{s} = \frac{s^2(t)}{s^2(t)+1} - \delta s + C, \qquad s(0) = s_0,$$

$$\dot{\psi}_i = \left[\delta + \rho - \frac{2s}{(s^2+1)^2}\right]\psi_i + 2\omega s$$

The transversality condition is $\lim_{t\to\infty} e^{-\rho t}\psi_i(t) = 0$.

[13] With $\delta = 1/2$ and $C = 0$, one obtains $h(s;0,1/2) = -(s/2)(s^2-2s+1)$. Then $h = 0$ at $s_L = 0$ and $s_M = s_H = 1$.

The symmetric open-loop Nash equilibrium is the solution of the following system of differential equations

$$\dot{s} = \frac{s^2(t)}{s^2(t)+1} - \delta s + C, \qquad s(0) = s_0,$$

$$\frac{\dot{C}}{C} = -\left[\delta + \rho - \frac{2s}{(s^2+1)^2}\right] + \frac{2\omega s C}{n},$$

with the transversality condition $\lim_{t\to\infty} e^{-\rho t}(n/C) = 0$. This system may possess multiple steady states, depending on parameter values.

It is useful to compare the open-loop Nash equilibrium and the social optimum. In the latter case, assume that a social planner maximizes the sum of the welfare of the n regions. This leads to the a different system of differential equations:

$$\dot{s} = \frac{s^2(t)}{s^2(t)+1} - \delta s + C, \qquad s(0) = s_0,$$

$$\frac{\dot{C}}{C} = -\left[\delta + \rho - \frac{2s}{(s^2+1)^2}\right] + 2\omega s C,$$

where we can see that the evolution of aggregate discharge is independent of the number of regions, n.

Comparing the two sets of differential equations, we notice that an open-loop Nash equilibrium with pollution damage parameter ω can be found by solving the optimization problem of a social planner who happens to have a lower damage parameter, say ω', where $\omega' = \omega/n$.[14]

To illustrate, consider the following parameter values: $\delta = 0.6, \omega = 1, \rho = 0.03$. Then the social planner's solution has a unique steady state, $s = 0.353$. It is stable in the saddlepoint sense. It can be shown that the social planner's optimal path of C is non-monotone when the initial level of pollution is large: C at first declines, then increases again, approaching the steady state level of discharge from below. On the other hand, when players do not cooperate, the open-loop Nash equilibrium has three steady states: an unstable one with a medium level of pollution, situated in between two saddle-point stable ones, $s_L = 0.398$ and $s_H = 1.58$.

Suppose that it is socially desirable to achieve the oligotropic steady state. Then, as expected, a time-dependent tax rate per unit of discharge can guide the open-loop players to achieve the socially optimal rate of discharge. Mäler et al. (2003) considered the restriction that the tax rate must be time-independent. They found for $n \leq 7$, under a suitably chosen time-independent tax rate, the phase diagram for the open-loop Nash equilibrium is qualititively similar to the phase diagram under a social planner, and the optimal steady state can be achieved, though welfare along the path toward the steady state will generally falls short of the welfare level that would be achieved under central control. However, for $n > 7$, the phase diagram for the open-loop Nash equilibrium under a fixed tax rate can be quite irregular, such that it may not be possible to guide the system to the socially optimal steady state.

[14] This property is crucially dependent on the special structure of the model, and on the assumed functional forms e.g. logarithmic utility.

We now turn to the symmetric Markov-perfect Nash equilibrium. The HJB equation for player i is

$$\rho V_i(s) = \max_{c_i} \left\{ \ln c_i - \omega s^2 + V_i'(s) \left[\frac{s^2}{s^2+1} - \delta s + c_i - (n-1) c_j(s) \right] \right\}$$

Using symmetry, the equilibrium feedback strategy must satisfy

$$c_i(s) = -\frac{1}{V'(s)} \equiv c(s)$$

Then, the HJB equation yields the identity

$$\rho V(s) = \ln c(s) - \omega s^2 - \frac{1}{c(s)} \left[\frac{s^2}{s^2+1} - \delta s + n c(s) \right] \text{ for all } s$$

Differentiating this identity, we obtain the following differential equation

$$\begin{aligned} & \left[\delta s - c(s) - \frac{s^2}{s^2+1} \right] c'(s) \\ = & \left(\rho + \delta - 2\omega s - \frac{2s}{(s^2+1)^2} \right) c(s) \end{aligned}$$

Since a closed form solution cannot be obtained, numerical solutions can be computed after specifying parameter values.[15] With $\delta = 0.6, \omega = 1$ and $\rho = 0.03$, it is found that the locus of possible steady states (in the space (s, c)) is non-monotone. As in the model of Dockner and Long (1993), there is a continuum of steady states, each corresponding to a feedback Nash equilibrium strategy. To each steady state stock s^* the corresponding individual emission level is

$$c^* = \delta s^* - \frac{(s^*)^2}{(s^*)^2+1}$$

The steady state $s = 0.38$ is of particular interest, because it can be reached only from $s_0 > 0.38$. It is also found that if $s_0 \in (0.17, 0.38)$, then strategies that start just a little above the point $(s_0, \delta s_0 - s_0^2/(s_0^2+1))$, will result in a state trajectory that converges to a steady state to the right. Steady state pollution stocks that are smaller than 0.17 are unstable. The important point is that pre-play communications allow the choice among feedback strategies, bringing the pollution stock closer to the social optimal one (0.38 is close enough to 0.353).

Note that the result of Dockner and Long (1993), that the best feedback equilibrium steady state is arbitrarily close to the social optimal one if the discount rate tends to zero, carries over to the lake-pollution model. Nevertheless, we should not forget that there is a distinction between welfare at a steady state, and total welfare, which takes into account the welfare flows along the path to the steady state.

Given the feedback information structure, it is natural to consider the design of efficiency-inducing taxation where the tax rate on emissions is made dependent only on the state

[15] For details, see Kossioris et al. (2008).

variable: $\tau = \tau(s)$. This issue was considered by Benchekroun and Long (1998) in the context of a polluting oligopoly. They found that there exists a feedback tax scheme that ensure that the oligopolists replicate the socially optimal path. In the context of lake-pollution, where the transition dynamics is more complicated, Kossioris et al. (2011) focus on polynomial functional forms for the tax rate $\tau(s)$. They found that it is not possible to completely mimic the social optimal path when the polynomial is of low order.

6 Dynamic Games involving Potential Regime Shifts and Skiba Point: R&D races and sabotage to prevent entry

In industrial organization theory, R&D races between firms have been a subject of intensive study. The winner of a race becomes a monopolist, so that there is a regime shift from, say, duopoly, to limit-pricing monopoly. Early models of R&D races assume that the time of a successful innovation is exponentially distributed: past investments in R&D have no strategic implications because the accumulated knowledge has no value (Loury, 1979, Lee and Wilde, 1980, Reinganum, 1982). This is because of the memorylessness of the exponential distribution, under which, if the event has not occurred, the future always looks the same, regardless of past levels of R&D. The resulting races cannot feature history dependence. The idea of a firm being ahead of another firm cannot be formulated under the assumption of exponential distribution. To capture the idea of history dependence, some authors propose multi-stage race models: to win a race, a firm must be the first to complete all stages of an R&D project. Thus, at any point of time, a firm may be ahead of another one. Several papers consider only deterministic multistage race models (Fudenberg et al., 1983; Harris and Vickers, 1985a, 1985b). In such models, the equilibrium result is drastic: once a firm has a slight advantage, the other firm drops out immediately. This is called the ε pre-emption property. To avoid this unrealistic feature, one could add the assumption that the stage-to-stage transition is probabilistic (Grossman and Shapiro, 1987; Harris and Vickers, 1987; Lippman and McCardle, 1987). However, these authors continue to assume that the time to completion for each stage is distributed exponentially. This implies that at each stage, firms' investment in R&D is independent of past investments. In these models, the laggard firm (that has completed fewer stages than its rival) will find it optimal to invest less than the industry leader, and consequently one observes that if a firm is behind, it tends to remain behind. This is not consistent with real world observations: there are instances of laggards' catching up behavior. To capture this catching up feature, Doraszelski (2003) formulates a model in which the hazard rate depends on both the state variable and the control variable.

Consider two firms. The stock of knowledge of firm i is denoted by $k_i(t)$, and its current R&D effort (a control variable) is denoted by $I_i(t)$. Doraszelski (2003) assumes that

$$\dot{k}_i(t) = I_i(t) - \delta k_i(t)$$

where $\delta > 0$ is the rate of depreciation of knowledge. The conditional probability that firm 1 makes a breakthrough over the interval of time $(t, t+\Delta)$, given that it has not been successful prior to time t is $h_1(I_1(t), k_1(t)) \times \Delta$, where the hazard rate function h_1 is specified as follows:

$$h_1(I_1, k_1) = \lambda I_1 + \gamma k_1^{\psi}$$

with $\lambda \geq 0$, $\gamma \geq 0$, and $\psi > 0$. Here, the hazard function $h_1(I_1, k_1)$ is additive and increasing in both the current effort, I_1, and the past efforts, as represented by k_1. (A multiplicative specification, e.g., $h_1 = I_1^{\alpha} k_1^{\beta}$, could be an interesting alternative, as Doraszelski (2003, p. 40) points out.) In the special case where $\gamma = 0$, past efforts do not influence the hazard rate, and we would obtain the memorylessness of the models of Reinganum (1981, 1982). The function h_1 is concave, linear, or convex in the state variable k_1 according to whether ψ is smaller than, equal to, or greater than 1. The cost of exerting effort I_1 is denoted by $c(I_1)$. This cost function is assumed to take the form

$$c(I_1) = \frac{1}{\eta} I_1^{\eta} \text{ where } \eta > 1.$$

Assume that as soon as one firm makes a breakthrough, the game ends, at which point the successful firm wins a big prize, $\overline{P} > 0$, and the other firm wins a small prize, $\underline{P} < \overline{P}$. The interpretation of these prizes is as follows. The high prize, $\overline{P}$, is the present value of future profits of the successful firm. The other firm can imitate the discovery after the patent has expired, and $\underline{P}$ is the present value of future profits of the imitating firm. The case where $\underline{P} = 0$ means that the innovating firm has perfect patent protection. In general, the ratio $\underline{P}/\overline{P}$ is a measure of patent protection. When this ratio is zero, the patent protection is perfect.

Denote the equilibrium strategies by $I_1 = \phi_1(k_1, k_2)$ and $I_2 = \phi_2(k_2, k_1)$. Then the HJB equation for firm 1 is

$$\begin{aligned} rW^1(k_1, k_2) &= \max_{I_1} \Big\{ \left[\overline{P} - W^1(k_1, k_2)\right] h_1(I_1, k_1) + \left[\underline{P} - W^1(k_1, k_2)\right] h_2(\phi_2, k_2) \\ &\quad - c_1(I_1) + \frac{\partial W^1}{\partial k_1}\left[I_1 - \delta k_1\right] + \frac{\partial W^1}{\partial k_2}\left[\phi_2 - \delta k_2\right] \Big\} \end{aligned}$$

The first order condition for I_1 is

$$\left[\overline{P} - W^1(k_1, k_2)\right] \lambda + \frac{\partial W^1}{\partial k_1} = c_1'(I_1) = I_1^{\eta - 1}$$

It follows that firm 1's strategy is

$$\phi_1(k_1, k_2) = \left\{ \left[\overline{P} - W^1(k_1, k_2)\right] \lambda + \frac{\partial W^1}{\partial k_1} \right\}^{\frac{1}{\eta - 1}}$$

Focusing on symmetric equilibrium, we can omit the subscript in the strategy functions ϕ_1 and ϕ_2 and the superscript in the value functions, W^1 and W^2, and thus we have

$$I_1^* = \phi(k_1, k_2) \text{ and } I_2 = \phi(k_2, k_1).$$

It follows that

$$\phi(k_1, k_2) = \left\{ \left[\overline{P} - W(k_1, k_2)\right] \lambda + \frac{\partial W(k_1, k_2)}{\partial k_1} \right\}^{\frac{1}{\eta - 1}}$$

$$\phi(k_2, k_1) = \left\{ \left[\overline{P} - W(k_2, k_1)\right] \lambda + \frac{\partial W(k_2, k_1)}{\partial k_2} \right\}^{\frac{1}{\eta - 1}}$$

and the HJB equation can be written as the operator equation

$$\mathcal{N}(W) = 0$$

where

$$\begin{aligned} \mathcal{N}(W)(k_1, k_2) \quad = \quad & \left(\lambda\phi(k_1, k_2) + \gamma k_1^{\psi}\right) \overline{P} + \left(\lambda\phi(k_2, k_1) + \gamma k_2^{\psi}\right) \underline{P} \\ & - \frac{\phi(k_1, k_2)^{\eta}}{\eta} - \left(r + \lambda\phi(k_1, k_2) + \gamma k_1^{\psi} + \lambda\phi(k_2, k_1) + \gamma k_2^{\psi}\right) W(k_1, k_2) \\ & + \frac{\partial W(k_1, k_2)}{\partial k_1} \left[\phi(k_1, k_2) - \delta k_1\right] + \frac{\partial W(k_1, k_2)}{\partial k_2} \left[\phi(k_2, k_1) - \delta k_2\right] \end{aligned}$$

This is a non-linear first-order partial differential equation. Since a closed form solution is not available, one must resort to numerical methods. Doraszelski (2003) reports the following numerical results.

(i) For the case where $\gamma = 0$ (i.e., the hazard rate is a function of current R&D effort only), the equilibrium R&D efforts are constant, independent of the knowledge stocks k_1 and k_2. The value function W is then a constant (independent of k_1 and k_2). This corresponds to the memoryless R&D race models (Reinagum, 1981, 1982).

(ii) When $\gamma > 0$, the accumulated knowledge stocks k_1 and k_2 matter. One can show that for any given finite k_2, $\lim_{k_1 \to \infty} W(k_1, k_2) = \overline{P}$, and for any given finite k_1, $\lim_{k_2 \to \infty} W(k_1, k_2) = \underline{P}$. With $\gamma > 0$, if $\lambda = 0$ (i.e., current R&D effort does not contribute directly to the hazard rate), the optimal R&D expenditure I_1 falls as the knowledge stock k_1 increases. That is, thanks to the "pure knowledge effect" on the hazard rate, the firm "can afford to scale back its investment in R&D as its knowledge stock increases" (p. 28).

(iii) When $\psi > 1$ so that h_1 is strictly convex and increasing in k_1, the increasing return to knowledge accumulation gives the firm a strong incentive to increase I_1 as k_1 rises from its low initial levels.[16] In particular, if firm 1 is a laggard (i.e., $k_1 < k_2$), it will try to catch up with firm 2 (i.e., investing more than firm 2) provided the gap between the two stocks is not too large. This catching-up feature is consistent with real world experience. Doraszelski (2003, p. 20) presented some evidence of catching up:

> "Casual observation suggests that the laggard strives to catch up with the leader. When Transmeta unveiled its power-stingy Intel-compatible Crusoe chip in 2000, Intel pledged to introduce a version of its Pentium III processor that matched Crusoe's power consumption in the first half of 2001 and announced a new set of technologies for 2002 or 2003 that would give it the lead over Transmeta. Similarly, after Celera Genomics in 1998 challenged the Human Genome Project to be the first to sequence the human genome, the Human Genome Project announced that it would move up its target date from 2005 to 2003 and indeed dramatically stepped up its own pace during 1999. And yet, although Celera Genomics started the race as the underdog, it completed a draft of the human genome in 2000 and beat the Human Genome Project."

[16] However, eventually when k_1 is large enough, I_1 begins to fall.

Doraszelski (2003) relied on the (ex-ante) symmetry between firms. Also, he did not attempt to explore the possibility of multiplicity of steady states and of Skiba points. As we have pointed out, the analysis of optimal control problems with multiple steady states involves the identification of a Skiba point. Skiba points can occur also in dynamic games. Dockner and Wagener (2014) give and example of Skiba point in a differential game invoving two symmetric players and a single capital stock. An interesting question is whether a Skiba point can exist when players are asymmetric. The paper by Dawid et al. (2016) presents a dynamic game model which exhibits the Skiba point property with two assymetric players and one capital stock.

Dawid et al. (2016) pointed out that general, it is unlikely to have a Skiba point when players are asymmetric, because it would require the existence of a point at which two assymetric players are indifferent between two courses of actions. Generically, it is impossible in an asymmetric game to have a single point where for each player, the two local value functions intersect. It follows that without very specific assumptions, it is unlikely that an MPE exhibiting Skiba points can exist. To illustrate this argument, consider an example discussed in Dawid et al. (2016). Suppose there are two firms, each investing in its own capital stock. Assume the firms produce goods that are perfect substitutes and they compete as Cournot rivals. Each firm would prefer that its rival invests less, because a low aggregate capital stock implies low aggregate output, which raises the price. *Given the strategy of firm 2*, suppose that a Skiba point, say k_1^S, exists for firm 1's optimal control problem. Then firm 2's value function would jump down as its rival's capital stock k_1 reaches k_1^S from below. Firm 2 therefore has an incentive to prevent firm 1's capital stock to get close to k_1^S, and thus it would want to "overinvest" (to deviate from the given candidate strategy) in order to induce firm 1 to invest less. Such optimal behavior by firm 2 would then implies that firm 1 would invest less even for values of $k_1(0)$ that are slightly above k_1^S, i.e., k_1^S cannot be a Skiba point. While this argument is intuitively plausible, nevertheless a formal analysis of a two-state-variable differential game between two asymmetric players that would establish the existence, or impossibility of existence, of a Skiba point is unfortunately unavailable.

Dawid et al. (2016) choose to work with a simpler model with two asymmetric players. They assume that there is only one stock of capital. There are two firms. The authors assume that firm 1, the incumbent firm, does not invest in R&D, and firm 2 is seeking to enter the market. Firm 2 can enter the market only if it is able to make a technological breakthrough. In order to make a breakthrough, firm 2 must invest in its stock of knowledge, k. If a breakthough has not occurred at time t, the probability that it will occur during the time interval $(t, t + dt)$ is given by $h(k(t))dt$. The function $h(k)$ is called the hazard rate. Dawid et al. (2016) assume that

$$h(k) = \alpha k^2, \; \alpha > 0.$$

This implies that there is increasing return to capital (in terms of probability of a breakthrough). They specify the following state dynamic equation,

$$\dot{k}(t) = I_2(t) - \lambda I_1(t) - \delta k(t)$$

where δ is the rate of depreciation, $I_2(t) \geq 0$ is firm 2's investment (R&D efforts), and $I_1(t) \geq 0$ is firm 1's sabotage effort. The positive parameter λ is a measure of the effectiveness of sabotage. The cost of I_i is $c_i(I_i) = \beta_i I_i + (\gamma_i/2)I_i^2$, with $\beta_i \geq 0$ and $\gamma_i \geq 0$.

A breakthrough by firm 2 implies a regime shift, from monopoly (under firm 1) to duopoly. Under duopoly, firm i earns a profit of π_i^d at each point of time. Under monopoly, $\pi_1 = \pi_1^m > 0$ and $\pi_2 = 0$. Assume that $\pi_1^m > \pi_1^d$, so that firm 1 has an incentive to sabotage firm 2's R&D efforts.

In order to establish the existence of a Skiba point, Dawid et al. (2016) find it neceesary to assume that there is an exogenous upper bound, denoted by $\overline{I}$, on I_i, $i = 1, 2$. This implies an upper bound on k: $k \leq \overline{k} = (1/\delta)\overline{I}$. The upper bound on investment is a crucial assumption, which results in a special property of the model: the value function of the incumbent is discontinuous at the Skiba point. The upper bound on sabotage makes it impossible for the incumbent to move the state variable k from the lower branch of its value function to the upper branch. (If the upper bound on the control were removed, so that any player could move the state in both directions, then the value function of each player would be continuous under the equilibrium profile.)

Formally, the dynamic game considered by Dawid et al. (2016) is a multi-mode game with two modes, m_1 (before entry) and m_2 (after entry), with $\pi_1(m_1) = \pi_1^m$, $\pi_2(m_1) = 0$, $\pi_1(m_2) = \pi_1^d$ and $\pi_2(m_2) = \pi_2^d$. (Dockner et al. (2000) refer to such multi-mode games as piece-wise deterministic game.) Firm i's objective is to maximize

$$J_i = E\left[\int_0^\infty e^{-rt}\left[\pi_i(m(t)) - c_i(I_i(t))\right] dt\right]$$

subject to $\dot{k}(t) = I_2(t) - \lambda I_1(t) - \delta k(t)$ and subject to the mode process

$$\lim_{\Delta \to 0} \frac{\Pr\left\{m(t+\Delta) = m_2 |\ m(t) = m_1\right\}}{\Delta} = h(k(t)),$$

with $m(0) = m_1$ and $k(0) = k_0$. Both firms set $I_i = 0$ in mode 2, while in mode 1 they use feedback strategies $I_i = \phi_i(k)$.

Clearly, in mode 2, the value functions are independent of k:

$$V_i(m_2) = (1/r)\pi_i^d.$$

Denote firm i's value function in mode 1 by $W_i(k)$. Then, in mode 1, the HJB equation for firm 1 is

$$rW_1(k) = \alpha k^2 \left[(1/r)\pi_1^d - W_1(k)\right] + \max_{I_1}\left[\pi_1^m - c_1(I_1) + W_1'(k)\left(\phi_2(k) - \lambda I_1 - \delta k\right)\right]$$

and the HJB equation for firm 2 is

$$rW_2(k) = \alpha k^2 \left[(1/r)\pi_2^d - W_2(k)\right] + \max_{I_2}\left[-c_2(I_2) + W_2'(k)\left(I_2 - \lambda\phi_1(k) - \delta k\right)\right]$$

Then the first-order condition for firm 1 is

$$\beta_1 + \gamma_1 I_1 = -\lambda W_1'(k) \text{ if } I_1 \in \left(0, \overline{I}\right)$$

and, for firm 2,

$$\beta_2 + \gamma_2 I_2 = W_2'(k) \text{ if } I_2 \in \left(0, \overline{I}\right).$$

Assuming that the equilibrium strategies are almost everywhere continuous on $\left[0, \overline{k}\right]$, and writing

$$\phi_1(k) = -\frac{\lambda W_1'(k)}{\gamma_1} - \frac{\beta_1}{\gamma_1}$$

$$\phi_2(k) = \frac{W_2'(k)}{\gamma_2} - \frac{\beta_2}{\gamma_2}$$

one obtains a system of two first-order differential equations for $W_1(.)$ and $W_2(.)$. Unfortunately, no closed form solution is available. The authors therefore resort to numerical methods. They use the homotopy method (see Vedenov and Miranda, 2001, for a discrete time model, Dawid et al., 2017, for a continuous time model).[17]

In the model, an increase in k has two qualitatively different and countervailing effects on the payoff of each player. First, since $h(k) = \alpha k^2$ is strictly convex and increasing, the effect of a marginal increase in k is more substantial at high levels of k. Therefore, a high k means a much greater chance of a regime switch. Second, a high k means the expected arrival time is closer to the present, which has the effect of reducing the impact of an increase in k on the expected future payoff stream of both players (bearing in mind that the size of k is irrelevant in mode 2, after entry). These opposing considerations suggest that equilibrium steady states with high and low investments for both players may co-exist.

Indeed, numerical calculations show that there are two locally stable steady states, one with high investment (or sabotage) by both players, one with low activities by both. The steady states are $k^* = 0$ and $k^{**} = 0.556$. There exists a Skiba point at $k^S < 0.556$.

7 Dynamic Games of Inducing Regime Shifts by a Big Push

Tornell (1997) presented a model of economic growth and decline with endogenous switches in property-right regimes when rival fractions incur a lumpy cost to overthrow an existing regime. In his model, two groups of infinitely-lived agents solve a dynamic game over the choice of property rights regime. He sought to find a possible equilibrium of the game involving multiple switching of regimes. Tornell allowed each group's share of aggregate capital to change after a switch takes place and introduced a once-off lump sum cost at switching time. Specifically, Tornell (1997) specified three property rights regimes: common property, private property, and leader-follower. Under common property, both players have equal access to the aggregate capital stock. When one player incurs the once-off cost, it can convert the whole common property to its private property unless the other player is willing to incur the same cost. In the latter case, the result is the private property regime, where each player has access only to its own capital stock. In contrast, starting from the private property regime, if both players simultaneously incur each the once-off cost, the regime will revert back to common property. If one player incurs the once-off cost while the

[17] This method yields polynomial approximations of value functions. One shortcoming is that such polynomial approximation gives continuous and smooth value functions, which may be incorrect. To deal with this issue, Dawid et al. (2016) combine the homotopy method with another method that yields local value functions (each around a stable steady state).

other does not, the former becomes the leader and has exclusive access to the economy's capital stock. Tornell (1997) restricted the maximum number of regime switches to two. This simplifying assumption allows closed-form solutions. A key parameter in this game is σ, the elasticity of intertemporal substitution. The model generates a hump-shaped pattern of growth even though the underlying technology is linear and preferences exhibit a constant elasticity of intertemporal substitution. If $\sigma \leq 1$, the common property regime may last for ever. (Alternatively, if the economy starts with the private property regime, this institution may also last for ever.) In constrast, if $\sigma > 1$, the economy exhibits a cycle: a switch from the common property regime to the private property regime, and later on, a re-switching back to common property. There is no equilibrium which involves a switch to the leader-follower regime.

While Tornell (1997) assumed that the two players are symmetric, Long et al. (2017) consider a model of regime-shift-inducing lumpy investments by asymmetric players. Each player can switch from one exploitation technology to another. They consider an economy that can operate under four possible regimes, denoted by I, N_1, N_2 and B. There are two players in this game, denoted by 1 and 2. Each player can make a big push only once during the game. Initially, the economy operates under regime I (where I stands for "initial"). Player 1 (he) can make a big push to switch the regime from I to N_1 which is to his advantage. However, player 2 (she) can pre-empt the rival's move by making a big push beforehand, thus switching the regime from I to N_2, to her advantage. In the case where both players make a big push at the same time, the economy's regime is switched from I to B (where B stands for "both"). Once the economy is in regime B, no further switch is possible. Regime B can also become operative after two consecutive big pushes, one by each player.

Let $\mathcal{S}$ denote the set of possible regimes, i.e.,

$$\mathcal{S} \equiv \{I, N_1, N_2, B\}.$$

Let $\mathcal{S}_i$ be the subset of regimes of $\mathcal{S}$ from which player i can make a Big Push. Then $\mathcal{S}_1 = \{I, N_2\}$ and $\mathcal{S}_2 = \{I, N_1\}$.

There is a continuous state variable, denoted by a vector $x \in \mathbb{R}_+^m$. For example, x is the economy's capital stock. To simplify the exposition, the authors set $m = 1$. In addition to a big push, each player also has a piece-wise continuous control variable c_i, with $c_i \in \mathbb{R}^n$. The instantaneous payoff $u_i(t)$ to player i at time t when the system is in regime $s \in \mathcal{S}$ is a differentiable function of the two control variables and the continuous state variable, and is in general different across regimes:

$$u_i(t) = U_i^s(c_i(t), c_{-i}(t), x(t)).$$

If player i, $i = 1, 2$, takes a regime change action at time $t_i \in \mathbb{R}_+$, he/she incurs a lumpy cost $K_i(x(t_i))$. If $0 < t_1 < t_2 < \infty$, the total payoff for player 1 is

$$\int_0^{t_1} U_1^I(c_1, c_2, x)e^{-rt}dt + \int_{t_1}^{t_2} U_1^{N_1}(c_1, c_2, x)e^{-rt}dt$$
$$+ \int_{t_2}^{\infty} U_1^B(c_1, c_2, x)e^{-rt}dt - K_1(x(t_1))e^{-rt_1}$$

with $r > 0$ is the discount rate.

The differential equation describing the evolution of the state variable x in regime s is

$$\dot{x} = G^s(c_1, c_2, x)$$

where, for each regime s, the function G^s is twice differentiable in the triplet (c_1, c_2, x).

For expositional purposes, Long at al. (2017) focus on a specific sequence of regimes: I, N_1 and B. A natural way to proceed, for determining a MPE of this game, is to solve the problem recursively, starting from regime B, the last regime of the system. Recall that each player has two types of controls, a piece-wise continuous control variable c_i, and a Big-Push date, t_i. A Markovian strategy consists of a control policy and a Big-Push rule at every possible state of the system, $(x, s) \in \mathbb{R}_+ \times \mathcal{S}$. The *control policy* of player i is a mapping $\eta_i(.)$ from the state space $\mathbb{R}_+ \times \mathcal{S}$ to the set $\mathbb{R}^n$. To get an idea of a *Big-Push rule*, consider the following situation. Suppose player 1 thinks that if player 2 finds herself in regime N_1 at date t, she will make a Big Push at a date $t_2 \geq t$. Then player 1 conjectures that the interval of time between the current period and the switching date, $t_2 - t$, is a function of the state of the system. More generally, define player i's *time-to-go (before making a Big Push),* given that $s \in \mathcal{S}_i$, as a mapping $\ell_2(.)$ from $\mathbb{R}_+ \times \mathcal{S}$ to $\mathbb{R}_+ \cup \{\infty\}$. For example, from the state (x, N_1), the real number $\ell_2(x, N_1)$ is the length of time that must elapse before player 2 makes her Big Push. If $\ell_2(x, N_1) = \infty$ for all x, this would mean that she does not want to make a Big Push if she finds herself under regime N_1.

Long et al. (2017) introduce the concept of piece-wise feedback Nash equilibrium (PFBNE), defined as follows:

(i) A strategy vector of player i is a pair $\chi_i \equiv (\eta_i, \ell_i)$.

(ii) A strategy profile (χ_1, χ_2) is a piece-wise feedback Nash equilibrium (PFBNE) if starting at any time t and any state (x, s), the remaining life-time payoff of player i is maximized by χ_i, given χ_{-i}.

As an application, Long et al. (2017) consider a game of exploitation of an exhasustible resources. There are two players. Each can choose a date at which she introduces a more efficient extraction technology. They find that the player with low investment cost is the first player to adopt a new harvesting technology. She faces two countervailing incentives: on the one hand, an early switch to a more efficient technology enables her to exploit the resources more cheaply; on the other hand, by inducing the regime change, which tends to result in a faster depletion, she might give her opponent an incentive to hasten the date of his technology adoption, if the opponent investment cost decreases as the stock decreases. As a consequence, in an equilibrium, the balance of these strategic considerations may make the low-cost player delay technology adoption even if her fixed cost of adoption is zero, contrary to what she would do (namely, immediate adoption) if she were the sole player.

Let us now contrast the Big-Push class of models (as considered in Tornell (1997) and Long et al. (2017)), with the other polar case where a regime shift can occur only with gradual investments. For illustration, we review the model of Itaya and Tsoukis (2019), who analysed a differential games involving symmetric agents who want to change their preferences away from envy-driven consumption. Itaya and Tsoukis (2019) considered a community consisting of n infinitely-lived agents who may contribute to the accumulation of a stock of "social capital", denoted by S. The higher is the stock, the lower is each individual's incentive to "out-do others" in terms of relative consumption. This incentive is captured by the term $(1-\theta_i(S)) \geq 0$, where $\theta_i(.)$ is an increasing function of S, with the property that $0 \leq \theta_i(S) \leq 1$ for all $S \geq 0$. The function $\theta_i(.)$ is the same for all i. Each individual i has 1 unit of time at each t. A fraction a_i of time is devoted to building up social capital (e.g., by spending time to socialize with other members of the community). The remaining fraction, $1-a_i$, is used to produce a consumption good, under the constant returns to scale technology $c_i = 1-a_i$. Production of c_i yields the utility of consumption, $\ln c_i$, from which the disutility of effort, βc_i, must be subtracted. The utility flow at time t is

$$\ln c_i(t) - \beta c_i(t) + (1-\theta_i(S(t))) \ln \left[\frac{c_i(t)}{C(t)/n}\right]$$

where C/n is the community's average consumption. While everyone knows that $\ln\left[\frac{c_i}{C/n}\right] = \ln(1) = 0$ in a symmetric equilibriun, it remains true that as long as $(1-\theta_i(S)) > 0$, each individual has an incentive to try to "out-do" others in terms of consumption, by spending a lot of time in production activities. This is the well-known "rat race" which reduces welfare. There is also an incentive to eliminate the rat race. If S is built up to the level $\overline{S}$ where $(1-\theta_i(\overline{S})) = 0$ and maintained at that level for ever, the rat race will be completely eliminated.

The authors assume that

$$\dot{S} = \left(\sum_{i=1}^{n} a_i\right) S - \delta S$$

where δ is the rate of depreciation of S.

The authors describe the set of Markov-perfect equilibria (MPEs) of this game. They show that there are a continuum of MPEs, which can either involve a monotone decreasing path $S(t)$, ending up at $S=0$, or a a monotone increasing path $S(t)$, ending up at $S=\overline{S}$. There is no stable equilibrium path that converges to an interior stock $S \in \left(0, \overline{S}\right)$.

8 Directions for Future Research

The literature on regime shifts has contributed much to our understanding of the complexity of the problems that decision makers face in a world where state dynamics are not immutable. We have learned from this literature that decision makers should be very cautious when they face uncertainty about tipping points. Development planning should take account of threshold externalities, and foreign aids could be more useful if donor countries can coordinate on a big push. Analysis of political changes can benefit from models of how discontent might build up. While the literature on regime shifts is indeed very rich, there are a number of issues that deserve greater scrutiny.

The first issue concerns the analysis of changes in preferences. While the existing literature acknowledges that preferences may change, typically such changes are either assumed to be exogenous (e.g., Kemp and Long, 1977), or triggered when an environmental threshold is crossed (e.g., Nkuiya and Costello, 2016), or comtemplated by infinitely-lived agents, as in Itaya and Tsoukis (2019). However, a more important class of actions should be considered: how to influence the preferences of the future citizens so that environmental thresholds can be managed more efficiently. The literature on social investments that affect preferences of future generations is sparse. For models of intergenerational transmissions of preferences, see Bisin and Verdier (2001, 2009, 2017) on the selection of traits, and Long (2019) on the moral education to encourage prosocial behavior, switching players' preferences from Nashian to Kantian.[18] As Bowles (2016) points out, a "moral economy" is more effective than an incentive-based economy in mitigating externalities and promoting investments in public goods.

The second issue concerns alternative paradigms for the analysis of regime shifts. Admittedly, the dominant paradigm in economic analysis is based on rational, forward looking behavior. However, evolutionary game theory has been successfully used to explain many phenomena.[19] It would be interesting to model regime shifts in human societies from an evolutionary perspective.

Acknowledgement: I would like to thank two anonymous reviewers for their careful reading of the paper and for their very helpful comments and suggestions.

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[18] On Nashian behavior versus Kantian behavior, see, e.g., Grafton et al. (2017).

[19] In a recent interesting paper, using evolutionary game theory, Wood et al. (2016) outline a model that explains well a regime shift in the world oil market: the "Seven Sisters" were replaced by OPEC in the battle for oil market dominance.

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