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Long-Run Market Configurations in a Dynamic Quality-Ladder Model with Externalities*

Mario Samano†, *Marc Santugini*‡

Résumé/Abstract

We analyze the type of market structures that arise in the long-run when quality externalities and asymmetric R&D capabilities exist in the context of a quality-ladder dynamic model. An example of such externalities is a patent release by the leading firm: an improvement of quality of this firm's good affects the quality of the other firms' products. This externality can be thought of as an increase in compatibility in a network. We show that it is possible for this model to generate, in the long-run, multi-modal probability distributions over different market structures from the same parameter values. In some cases, the lagging firm may even become the dominant firm depending on the degree of the externality. This may have implications for the estimation and simulation of this class of models.

Mots clés/Key words: Dierentiated-good markets, Quality-ladder model, Externalities, Industry dynamics, Market structures.

Codes JEL/JEL codes: C61, C73, L13

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1 Introduction

We study situations in which the leading firm causes a positive externality on the quality of the good produced by its competitors. For instance, the decision of a firm to release its patents in order to increase the overall market share of the industry by increasing the degree of compatibility among all competitors. Recently, this has been the strategy taken by Tesla Motors, the manufacturer of electric vehicles (EVs). Its technology for electricity storage allows the Tesla Model S to have a range of around 200 miles per battery charge, significantly above that of its competitors, and their technology used in their network of recharging stations allows Tesla car owners to fully recharge the battery in about one hour, much below that of its competitors. In June 2014, Elon Musk, the CEO of Tesla Motors, announced that they were releasing most of the company's patents because he said "We believe that Tesla, other companies making electric cars, and the world would all benefit from a common, rapidly-evolving technology platform."[1] Patent releasing changes the trade-off between market dominance (of the leader releasing its patents) and overall industry growth (currently the market share of EVs is very small). Most likely, what the CEO of Tesla Motors expects is that the industry converges to the same technology standard for recharging stations so that the network expands for all EV users, which will increase the market share of EVs, including Tesla Motors' own market share.

In a more general framework, we think of this patent release strategy as a case of a positive externality on the quality of the competitors. Our research goal is to understand how the presence of this positive externality −due either to a firm's unilateral decision or to regulation− affects the leading firm and the industry overall. To that end, we model the difference between the leading and lagging firms by assuming that the return to investment in quality differs across firms. That is, for a given level of investment, one firm has a higher probability to raise the quality of the good it produces. We show that such a model can generate different types of long-run market configurations (market collapse, market dominance by *either* firm, duopoly, and combinations of these cases). We find the array of possible market structures that can arise from this game for different parameter values, including the case of dominance by the lagging firm when we consider a patent release situation modeled as an externality on quality proportional to the competitor's quality,

Even in the absence of positive externalities on quality, firms have different likelihoods

[1] https://www.tesla.com/blog/all-our-patent-are-belong-you?

of success of investment. Recently, Goettler and Gordon (2011) have estimated a dynamic quality-ladder model for the computer processors industry. They find evidence for heterogeneity in the likelihood of success of investment, which can explain differences in the levels of investment and ultimately differences in the levels of quality between the goods.[2] Motivated by this finding, we ask the following question. What is the effect of heterogeneity in firms' ability to invest in quality on long-run market configurations? This heterogeneity can be initially driven by intrinsic characteristics of the firms or by a quality externality such as a patent release action. To answer this question, we adapt the quality ladder model described in Ericson and Pakes (1995) and the algorithms to numerically solve for its equilibrium such as the one described in Pakes and McGuire (1994) and in a particular case in Levhari and Mirman (1980) to the case of heterogeneous likelihood of success of investment.

Another example of estimation of this class of models is Gowrisankaran and Town (1997). They consider two types of hospitals, for-profits and non-profits. The ratio of the number of these two types of hospitals is endogenous in their model. The parameter governing the probability of success of investment is restricted to be the same for the two hospital types, and yet, the observed market configurations in the data are not symmetric.[3]

We restrict attention to the quality-ladder model without entry or exit. This is not a strong assumption since we allow for quality levels of zero which in turn yield to zero demand, meaning that the firm producing such good acts as if it had exited the market. That however does not prohibit the same firm to become active again if it achieves to increase quality to a positive level in the next period. We also note that in our motivating example from Goettler and Gordon (2011), they do not consider entry and exit since the industry they study does not exhibit such behavior during the time window in their data.[4] In our second motivating example on the estimation of a quality-ladder model, Gowrisankaran and Town (1997) consider the possibility of entry and exit, however all hospitals belong to one of two firm types, and thus if all firms of one type exit, this is equivalent in our two-firm model to having quality zero for one type of firm.

[2] In their adaptation of the Ericson-Pakes model, the source of this heterogeneity in the model is twofold: specific parameters for each firm and the quality distance between the leader and the follower. Specifically, they find a parameter value for the likelihood of success of investment of 0.0010 for Intel and 0.0019 for AMD. The estimated parameters are different for each firm, which captures the observed heterogeneity of firm dominance in their data.

[3] In the data, this ratio was observed to be 16, meaning that the for-profits hospitals dominate the market. That parameter of the likelihood of success of investment is estimated to be 0.51, well within our parameter space specification.

[4] Goettler and Gordon (2011) pp. 1151.

Heterogeneity in the quality-ladder dynamic models has been studied in the context of capacity games. Besanko and Doraszelski (2004) conclude that asymmetries of firm size can be due to the effects of price competition which in turn lead to long run distributions that exhibit positive probabilities on outcomes that represent only one firm surviving.[5] Their analysis keeps parameters symmetric across the two firms. We also find such configurations in cases of symmetric firms, but those configurations can arise from other parameter combinations as well. The asymmetries in price competition in their model arise because of small asymmetries in capacity accumulation that occur accidentally which makes one firm slightly dominant over the other, making the other firm to give up if investment is highly reversible. In Borkovsky et al. (2010) and Borkovsky et al. (2012), it is shown that the dynamic quality-ladder model can exhibit multiplicity of equilibria even in the absence of entry or exit if the investment is highly permanent. We take a different approach and allow firms to have different parameters in their investment success function and study the long-run distribution over the quality space given the unique equilibrium policies.[6] We abstract from collaborations in R&D that could also lead to externalities.[7]

Our analysis shows that the dynamic quality-ladder model for two firms can generate, in the long-run, different probability distributions over the space of market configurations depending on parameter values and that in some cases these distributions are multi-modal. If that is the case, we argue that this can be interpreted as a positive probability that more than one market structure is possible for the same set of parameter values. We assess how each of the model parameters affects the results.

We find that asymmetries in the likelihood of success of investment can have relevant effects on long-run market configurations which shows the richness of the baseline model. We also find that changes in the depreciation rate can significantly affect the number and types of long-run market configurations. More specifically, the presence of higher depreciation rates increases the likelihood of market collapse and market dominance at the expense of the probability of duopoly. In our analysis of the model with externalities, we find that even though the externality may be beneficial to decrease the outside good market share, it could harm the leader and allow the lagging firm to dominate the market if the asymmetry in the

[5] This behavior was not found under quantity competition.

[6] In Figure 5 from Borkovsky et al. (2010), they provide evidence on the existence of multiple equilibria for depreciation rates below 0.1. Our analysis uses depreciation rates above or equal to that level and we check for potential multiplicity of equilibria solving the game in consecutive finite time horizons versions of the model a la Levhari and Mirman (1980).

[7] See Samano et al. (2017).

externality is above certain level and the return to investment is relatively low. Therefore, releasing patents can eliminate the advantage of the leader for a certain range of industry parameters. This is utterly important since it shows that sharing knowledge to competitors in the context of quality externalities is not always harmful.

This paper also has implications for the simulation of this type of models. Typically, one obtains data from an industry, say in a duopoly, and assume this is the equilibrium. Then a set of parameters is obtained using a dynamic model of competition that belongs to the class of models we analyze here. We show that it is possible that when simulating the industry with the estimated parameters, additional market structures may arise in the long-run. If we only report expected values for the different outcomes of the model, it is possible that salient information is being masked since the multiplicity of modes in the probability distribution may be eroded.

The remainder of this article has the following structure. Section 2 introduces the model. In Section 3 we provide computational details, the parametrization of the model, and the long-run distributions. Section 4 presents the main results. We discuss further connections to the literature and conclude in Section 5.

2 Model

In this section, we extend the Ericson-Pakes dynamic quality-latter model to the case in which each firm's valuation of the good sold depends not only on its own quality level, but is also potentially influenced by the quality level achieved by the other firm. For instance, in the case of the electric car industry, quality refers to the availability and effectiveness of the network of recharging stations. An improvement in the size or the effectiveness of this network translates into an improvement in quality and thus an increase in consumers' valuation for electric cars. There are two possible cases to study.

1. *No externalities.* Suppose that there is no compatibility among the different firms. Then, an improvement in the quality of one firm affects only consumers' valuation for its own good.

2. *Externalities.* Suppose that there is imperfect compatibility. For instance, one firm's electric cars can recharge in any recharging stations. Then, an improvement in the quality of one firm's network of recharging stations (i.e., a higher quality) affects (asymmetrically) consumers' valuation for all goods in that industry.

To study the long-run implications of such an industry, we must take account of heterogeneity. There are two kinds of heterogeneity worth considering.

1. *Quality externality.* The first layer of heterogeneity concerns the link between quality and consumers' valuation. For instance, the leading firm might not benefit from quality improvement of the lagging firm as much as the lagging firm would benefit from quality improvement on the part of the leading firm. This is represented by the parameter $\kappa_.$ below.

2. *Likelihood of success of investment.* The technological ability to improve quality varies across firms, i.e., some firms are more capable than others of turning investment into a successful upgrade in quality. This is represented by the parameter $\alpha_.$ below.

We now provide a detailed description of the Ericson-Pakes dynamic quality latter model under the presence of these two sources of heterogeneity. For simplicity, we restrict attention to the case of two firms and abstract from entry or exit.[8]

Demand. We consider a differentiated-product market in which two firms compete à la Bertrand as well as invest to improve the quality of their products. For $j = 1, 2$, let $\omega_j \in \{0, 1, 2, ..., M\}$ be firm j's quality of the product out of M possible values. Given qualities $\{\omega_1, \omega_2\}$ and prices $\{p_1, p_2\}$, firm j's demand is

$$D\left(p_j, p_{3-j}; \omega_j, \omega_{3-j}\right) = m \frac{e^{g_j(\omega_j, \omega_{3-j}) - \lambda p_j}}{1 + e^{g_j(\omega_j, \omega_{3-j}) - \lambda p_j} + e^{g_{3-j}(\omega_{3-j}, \omega_j) - \lambda p_{3-j}}}$$

where $m > 0$ is the size of the market and

$$g_j(\omega_j, \omega_{3-j}) = \begin{cases} -\infty, & \omega_j + \kappa_j \omega_{3-j} \leq 0 \\ \omega_j + \kappa_j \omega_{3-j}, & 1 \leq \omega_j + \kappa_j \omega_{3-j} < \omega^* \\ \omega^* + \log(2 - \exp(\omega^* - \omega_j - \kappa_j \omega_{3-j}), & \omega^* \leq \omega_j + \kappa_j \omega_{3-j} \leq M \end{cases} \tag{1}$$

maps firm j's product quality into consumer's valuation. The parameter $\omega^* \in (0, M]$ reflects the level of quality after which there is a degree of satiation.[9] Expression (1) introduces the type of heterogeneity in the model via the externality in quality. That is, consumers' valuation

[8] As discussed in the introduction, one of our two empirical examples in the literature (Goettler and Gordon (2011)) does not consider entry or exit. Moreover, we allow for quality levels of zero and the demand function in this case becomes null, this is equivalent to exiting the market.

[9] The last two lines in the specification of the function g reflect a satiation effect at high quality levels (above ω^*). This creates decreasing marginal returns on the utility function. The derivative of g is the same from the left and from the right of ω^*.

for good j depends on the quality achieved by firm $3-j$, i.e., ω_{3-j}. The parameter κ_j governs the influence of firm $3-j$. If $\kappa_1 = \kappa_2 = 0$, we obtain the baseline model without externality.[10] When $\kappa_{\cdot} > 0$ and increases, it reflects a greater positive influence of the competitor's quality on the firm's good. Differences between κ_1 and κ_2 mean that one firm benefits more from the quality externality than the other one. If $\kappa_j < 0$ there is a negative effect on the quality of good j from quality improvements from the other firm. Since our main motivation in this paper is a patent release, we concentrate on the case of positive externalities.

Profits. Firm j's instantaneous profits are

$$\pi\left(p_j, p_{3-j}; \omega_j, \omega_{3-j}\right) = D\left(p_j, p_{3-j}; \omega_j, \omega_{3-j}\right)\left(p_j - c\right)$$

where $c > 0$ is the constant marginal cost of production, same across firms. Because market competition has no effect on the dynamics, the pricing game is static. Let $\Pi\left(\omega_j, \omega_{3-j}\right)$ be firm j's instantaneous profit corresponding to the static Bertrand game.[11]

Investment. Each period, firm j invests an amount $x_j \geq 0$ intended to improve product quality. The process for quality is stochastic and subject to an industry-wide shock. Specifically, firm j's product quality evolves stochastically as

$$\omega_j' | \omega_j = \min\{\max\{\omega_j + \tau_j + \eta, 0\}, M\}$$

where τ_j is a firm-specific shock and η is an industry-wide depreciation shock. Each random variable is binary. The firm-specific shock η has support $\{0, 1\}$ and depends on the amount of investment, i.e.,

$$\Pr(\tau_j = 1 | x_j) = \frac{\alpha_j x_j}{1 + \alpha_j x_j} \equiv \phi_j(x_j)$$

is firm j's probability of success conditional on investing $x_j \geq 0$. Here, $\alpha_j > 0$ is specific to firm j, which is our second source of parameter heterogeneity. The industry-wide depreciation shock has support $\{-1, 0\}$ such that

$$\Pr(\eta = -1) = \delta \in [0, 1]$$

[10] Note also that our specification in (1) is similar to Borkovsky et al. (2012) in that $\omega_j = 0$ drives firm j's demand to zero. Although entry or exit are not explicitly modeled, the state $(\omega_1, \omega_2) = (0, 0)$ essentially leads to a temporary collapse of the market. We call it a temporary collapse since firms can still successfully invest in the next period to go back into the game. In other words, it is possible that for a particular set of parameters even if $(\omega_1, \omega_2) = (0, 0)$, firms' optimal policy functions are positive at that state.

[11] That is, for $j = 1, 2$, $\Pi\left(\omega_j, \omega_{3-j}\right) = D\left(p_j^*, p_{3-j}^*; \omega_j, \omega_{3-j}\right)\left(p_j^* - c\right)$ where the pair $\{p_1^*, p_2^*\}$ is the Bertrand equilibrium defined as $p_j^* = \arg\max_{p_j > 0} D_j\left(p_j, p_{3-j}^*; \omega_j, \omega_{3-j}\right)\left(p_j - c\right)$. For all $\{\omega_1, \omega_2\}$, there exists a unique Bertrand-Nash equilibrium (Caplin and Nalebuff (1991)).

is the probability of quality depreciation.[12]

Value Function. Before proceeding with the definition and characterization of the equilibrium, it is useful to write down the firm's value function taking as given the behavior of the other firm. Specifically, for $j = 1, 2$, given x_{3-j}, firm j's infinite-horizon value function satisfies

$$v_j\left(\omega_j, \omega_{3-j}\right) = \max_{x_j \geq 0} \left\{ \Pi\left(\omega_j, \omega_{3-j}\right) - x_j + \beta \mathbf{E}[v_j(\omega_j', \omega_{3-j}') | \omega_j, \omega_{3-j}, x_j, x_{3-j}] \right\}$$

where a prime sign indicates a variable in the subsequent period and the expected continuation value function is written as

$$\begin{aligned}
&\mathbf{E}[v_j(\omega_j', \omega_{3-j}') | \omega_j, \omega_{3-j}, x_j, x_{3-j}] \\
&= \phi_j(x_j)\phi_{3-j}(x_{3-j}) \cdot \left(\delta v_j\left(\omega_j, \omega_{3-j}\right) + (1-\delta) v_j\left(\omega_j^+, \omega_{3-j}^+\right)\right) \\
&\quad + \phi_j(x_j)(1-\phi_{3-j}(x_{3-j})) \cdot \left(\delta v_j\left(\omega_j, \omega_{3-j}^-\right) + (1-\delta) v_j\left(\omega_j^+, \omega_{3-j}\right)\right) \\
&\quad + (1-\phi_j(x_j))\phi_{3-j}(x_{3-j}) \cdot \left(\delta v_j\left(\omega_j^-, \omega_{3-j}\right) + (1-\delta) v_j\left(\omega_j, \omega_{3-j}^+\right)\right) \\
&\quad + (1-\phi_j(x_j))(1-\phi_{3-j}(x_{3-j})) \cdot \left(\delta v_j\left(\omega_j^-, \omega_{3-j}^-\right) + (1-\delta) v_j\left(\omega_j, \omega_{3-j}\right)\right)
\end{aligned} \tag{2}$$

with

$$\omega_j^+ \equiv \min\{\omega_j + 1, M\}, \tag{3}$$

$$\omega_{3-j}^+ \equiv \min\{\omega_{3-j} + 1, M\}, \tag{4}$$

$$\omega_j^- \equiv \max\{\omega_j - 1, 0\}, \tag{5}$$

$$\omega_{3-j}^- \equiv \max\{\omega_{3-j} - 1, 0\}. \tag{6}$$

Given an initial state (ω_j, ω_{3-j}) , expression (2) summarizes all possible changes in the states corresponding to investment levels (x_j, x_{3-j}).

Equilibrium. We restrict attention to Markov-perfect equilibrium (MPE) in pure strategies. The pair $\{X_1\left(\omega_1, \omega_2\right), X_2\left(\omega_2, \omega_1\right)\}$ is an equilibrium if, for $j = 1, 2$, given $X_{3-j}(\omega_{3-j}, \omega_j)$

$$\begin{aligned}
X_j(\omega_j, \omega_{3-j}) = \arg\max_{x_j \geq 0} \{ &\Pi\left(\omega_j, \omega_{3-j}\right) - x_j \\
&+ \beta \mathbf{E}[V_j(\omega_j', \omega_{3-j}') | \omega_j, \omega_{3-j}, x_j, X_{3-j}(\omega_{3-j}, \omega_j)] \}
\end{aligned}$$

[12]The specific values for α_j we use in our simulations lie well within those in the literature (Goettler and Gordon (2011), Gowrisankaran and Town (1997), Borkovsky et al. (2010)).

where for any $(\omega_j, \omega_{3-j}) \in \{0, 1, ..., M\}^2$, the value function satisfies

$$V_j(\omega_j, \omega_{3-j}) = \Pi(\omega_j, \omega_{3-j}) - X_j(\omega_j, \omega_{3-j}) \tag{7}$$
$$+ \beta \mathbf{E}[V_j(\omega_j', \omega_{3-j}')|\omega_j, \omega_{3-j}, X_j(\omega_j, \omega_{3-j}), X_{3-j}(\omega_{3-j}, \omega_j)] \tag{8}$$

where $\mathbf{E}[V_j(\omega_j', \omega_{3-j}')|\omega_j, \omega_{3-j}, X_j(\omega_j, \omega_{3-j}), X_{3-j}(\omega_{3-j}, \omega_j)]$ has the same form as equation (2).

From the first order condition we obtain for $j = 1, 2$,

$$X_j(\omega_j, \omega_{3-j}) = \max\left\{ \frac{-1}{\alpha_j} + \sqrt{\frac{\beta}{\alpha_j}} \sqrt{\frac{\alpha_{3-j} X_{3-j}(\omega_{3-j}, \omega_j)\Delta_j + \Psi_j}{1 + \alpha_{3-j} X_{3-j}(\omega_{3-j}, \omega_j)}}, 0 \right\} \tag{9}$$

when $\frac{\alpha_{3-j} X_{3-j}(\omega_{3-j}, \omega_j)\Delta_j + \Psi_j}{1 + \alpha_{3-j} X_{3-j}(\omega_{3-j}, \omega_j)} \geq 0$ and $X_j(\omega_j, \omega_{3-j}) = 0$ otherwise. Here, using (3) - (6),

$$\begin{aligned}
\Delta_j &\equiv \delta \left[V_j(\omega_j, \omega_{3-j}) - V_j\left(\omega_j^-, \omega_{3-j}\right) \right] \\
&\quad + (1-\delta)\left[V_j\left(\omega_j^+, \omega_{3-j}^+\right) - V_j\left(\omega_j, \omega_{3-j}^+\right) \right], \\
\Psi_j &\equiv \delta \left[V_j\left(\omega_j, \omega_{3-j}^-\right) - V_j\left(\omega_j^-, \omega_{3-j}^-\right) \right] \\
&\quad + (1-\delta)\left[V_j\left(\omega_j^+, \omega_{3-j}\right) - V_j(\omega_j, \omega_{3-j}) \right].
\end{aligned}$$

2.1 Strategic Complementarity and Substitutability in Investment

From the first order condition we can also obtain the rate at which firm j's investment changes with respect to the other firm's level of investment:

$$\left.\frac{\partial X_j}{\partial X_{3-j}}\right|_{(\omega_j, \omega_{3-j})} = \frac{\beta \frac{\alpha_{3-j}}{(1+\alpha_{3-j} X_{3-j}(\omega_{3-j}, \omega_j))^2}(\Delta_j - \Psi_j)}{2(1 + \alpha_j X_j(\omega_j, \omega_{3-j}))}.$$

All the factors in that expression are always positive except for the term $\Delta_j - \Psi_j$. Therefore, we would observe strategic substitutes in investment if $\Delta_j - \Psi_j < 0$ and complements if $\Delta_j - \Psi_j > 0$. The sign of this term depends on the degree of concavity of V, which in turn depends on its initial condition: the static profits, which is the only object that contains the information on the externality. We are unable to determine the exact mechanism by which the level of the externalities changes the curvature of this function but as explained in Section 4, we do not find evidence for strategic complements. This also suggests that there cannot be multiple solutions to the investment problem at each time period since the two reaction curves are decreasing and therefore they can only intersect each other at most once. If both reaction curves were increasing and concave, it would be possible to have more than one intersection. A simple inspection of the second derivative of the reaction function indicates that this could occur if α_j is very large and $\Delta_j - \Psi_j < 0$.[13]

3 Computation and Parametrization

We use the Pakes-McGuire (PM) algorithm to numerically solve for $\{X_1(\omega_1,\omega_2), X_2(\omega_2,\omega_1)\}$ and $\{V_1(\omega_1,\omega_2), V_2(\omega_2,\omega_1)\}$. Since firms can be heterogeneous, i.e., $\alpha_1 \neq \alpha_2$ and $\kappa_1 \neq \kappa_2$, the algorithm consists of iterating on the best response operators until convergence is reached. Specifically, at the initial iteration $\tau = 0$, we set

$$\{X_1^0(\omega_1,\omega_2), X_2^0(\omega_2,\omega_1)\} = \{0,0\},$$

for all combinations of states (ω_1,ω_2) and the corresponding value functions

$$\{V_1^0(\omega_1,\omega_2), V_2^0(\omega_2,\omega_1)\} = \{\Pi(\omega_1,\omega_2), \Pi(\omega_1,\omega_2)\}.$$

For iteration $\tau = 1,2,...,$ given $\{X_1^{\tau-1}(\omega_1,\omega_2), X_2^{\tau-1}(\omega_2,\omega_1)\}$ and $\{V_1^{\tau-1}(\omega_1,\omega_2), V_2^{\tau-1}(\omega_2,\omega_1)\}$, we construct X_1^τ and X_2^τ according to the reaction function given by (9) where the policy functions on the right hand side of that equation and the terms that depend on value functions are all indexed by the previous time period, that is $X_j^{\tau-1}$ and $V_j^{\tau-1}$.

Moreover, the value functions are defined by equation (7) and updated as follows:

$$V_1^\tau(\omega_1,\omega_2) = \Pi(\omega_1,\omega_2) - X_1^\tau(\omega_1,\omega_2) + \beta\mathbf{E}[V_1^{\tau-1}(\omega_1',\omega_2')|\omega_1,\omega_2,X_1^{\tau-1}(\omega_1,\omega_2),X_2^{\tau-1}(\omega_2,\omega_1)],$$
$$V_2^\tau(\omega_2,\omega_1) = \Pi(\omega_2,\omega_1) - X_2^\tau(\omega_2,\omega_1) + \beta\mathbf{E}[V_2^{\tau-1}(\omega_2',\omega_1')|\omega_2,\omega_1,X_2^{\tau-1}(\omega_2,\omega_1),X_1^{\tau-1}(\omega_1,\omega_2)].$$

The algorithm stops when some convergence criterion for the value functions and the policy functions is met.

In the PM algorithm, the computed levels of investment at each iteration do not necessarily constitute an equilibrium since the best responses (in terms of investment) at iteration τ are in reaction to the investments computed at iteration $\tau - 1$. However, stationary points of such iterations are MPEs. In addition to the PM algorithm, we also apply the algorithm suggested by Levhari and Mirman (1980) (LM) in a resource extraction dynamic game. The algorithm consists of computing the equilibrium for any finite horizon and increasing the

[13]By implicit differentiation, $X_j'' = -\frac{\alpha_j X_j'^2 + \alpha_k \beta(\Delta_j - \Psi_j)\frac{1}{(1+\alpha_{3-j}X_{3-j})^3}}{1+\alpha_j X_j}$.

horizon (making use of the computation for shorter horizons) until convergence is met. Unlike the PM algorithm, the levels of investment computed under the LM algorithm at each iteration constitute a Markov-perfect equilibrium. In our numerical analysis, we compute the equilibrium using both algorithms, which always lead to the same converged policy functions. The algorithm that computes the limit of a finite horizon game has been applied in the context of the Ericson-Pakes framework Goettler and Gordon (2011) and in Chen et al. (2009). A description of the LM algorithm is relegated to the Online Appendix. We note that the PM algorithm is much faster than the LM algorithm. However, the latter allows us to guarantee that the reaction functions cross at most once which suggests uniqueness of equilibrium. We discuss this further in the next two sections.

We use the same parameter values as in Borkovsky et al. (2010) except for the following: (i) the introduction of the externality κ is parametrized as fraction of the competitor's quality level, (ii) we use a slightly higher value for the price sensitivity λ, we use 1.2 and 1.7 instead of 1 simply to obtain a wider range of variation in long-run outcomes over our interval for $\alpha_{\cdot}$, equivalently we could lower the value of λ and increase the range of α.

Table 1: Parameter values

Parameter	M	m	c	ω^*	β	λ	α_j	κ_j
value(s)	18	5	5	12	0.925	$\{1.2, 1.7\}$	[0.1, 5]	[0, 1]

3.1 Long-run Distributions

Let $\mathbf{a}_t = [a_{t,0}, \ldots, a_{t,(M+1)^2}]$ be a vector of size $1 \times (M+1)^2$ where $a_{t,s}$ is the probability that the industry is in state $s = (\omega_j, \omega_k)$ at time t such that $\sum_s a_{t,s} = 1$.

Let $\mathbf{P}$ be a $(M+1)^2 \times (M+1)^2$ transition matrix such that each element provides the probability to transition from one industry state to another, i.e., $\Pr[(\omega_j', \omega_k') | (\omega_j, \omega_k)]$.[14] We can obtain the transient distribution at each time t by using the sequence

$$\mathbf{a}_t = \mathbf{a}_{t-1}\mathbf{P} \tag{10}$$

and therefore

$$\mathbf{a}_t = \mathbf{a}_0\mathbf{P}^t$$

[14]The Online Appendix provides a detailed derivation of the transition matrix.

where $\mathbf{a}_0$ is the initial distribution over the state space. For a given set of parameters, we obtain the converged policy functions $X^*(\omega_j, \omega_k)$ and use them to calculate $\mathbf{P}$. When this matrix has only one left eigenvalue equal to one, the limiting distribution $\mathbf{a}^*$ exists and satisfies

$$\mathbf{a}^* = \mathbf{a}^*\mathbf{P},$$

which guarantees that the limiting distribution is independent of the initial condition $\mathbf{a}_0$ and there is only one recurrent class. However, for some parameter values, we find two or more recurrent classes by using the sequence given in (10). Therefore, all of our results are obtained by using this sequential method starting from a uniform distribution over the state space. We stop when a converging criterion is met, refer to this distribution as the long run distribution $\mathbf{a}^*$. We also check that we obtain the same distribution if $\mathbf{a}_0$ is a degenerate distribution and we do not find large qualitative differences in our results with respect to a uniform distribution initial condition. Only for small regions of the parameter space for α do we find discrepancies in the number of modes of the long run distribution. It is worth remarking that other studies of this type of models have also opted for considering the long-run transient distributions instead of the limiting distributions (see Borkovsky et al. (2010) Section 5.3).

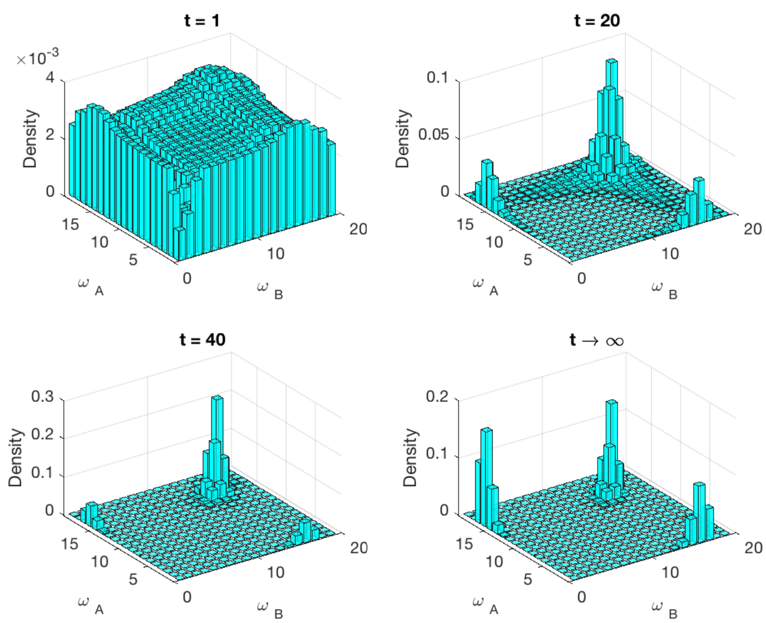
Once we obtain the distribution $\mathbf{a}^*$, we reshape this vector into an $(M + 1) \times (M + 1)$ matrix $\tilde{\mathbf{a}}$ and we count the number of modes. Each of these modes represents the maximum probability of a specific market configuration.[15] We will treat this distribution over the space of quality combinations as the full characterization of the market structures for a given set of parameters. It is worth discussing how this is equivalent to reporting the shapes of the distributions for the market shares for each firm. Equation (1) guarantees that all the values for the market shares are positive except when quality is zero. When we multiply element by element the matrix of market shares over the (ω_A, ω_B) space for firm A times $\tilde{\mathbf{a}}$ we obtain the same number of modes as in $\tilde{\mathbf{a}}$ except if there was a mode along the $q_A = 0$ line. If there was such a mode, it will appear as a mode on the element by element product of the matrix of market shares for B and $\tilde{\mathbf{a}}$.

In what follows we show that $\tilde{\mathbf{a}}$ might be unimodal (i.e., only one configuration occurs) or bimodal (two different market configurations are possible) or tri-modal (three different market structures can arise from the same set of parameters). Specifically, the market may

[15]We discard modes that have an associated probability of less than 10^{-3}. This threshold is equivalent to discard market structures that have an associated probability of less than 0.1% chance of occurring.

collapse, i.e., quality is driven to zero with probability one and firms do not sell anything. It is also possible to observe a duopoly. Finally, one firm may end up dominating, i.e., one firm offers a good of positive quality, i.e., $\omega_j \neq 0$ whereas the other firm offers a good of zero quality, essentially becoming insignificant, i.e., $\omega_{3-j} = 0$. For this case, it is possible to observe a realization in which the lagging firm dominates the market when there are externalities. Our main objective is to show that for a given set of parameter values, the resulting long-run distribution may imply a non-negligible probability for *different* market structures. Figure 1 shows an example in which $\tilde{\mathbf{a}}$ has three modes.

Figure 1: Transient distributions



Notes: Transient distributions from same policy function at different time periods. Initial distribution $\mathbf{a}_0$ is uniform.

In the next two sections, we provide a numerical analysis of the effect of heterogeneity on the long-run market structures. Also, to keep notation more tractable, instead of referring to firms by j and $3-j$, we will refer to them as A and B. We begin with the case of no quality externality, i.e., $\kappa_A = \kappa_B = 0$ so that an improvement of quality of firm A has no effect on consumers' valuation for the good sold by firm B. In that case, we investigate how an advantage in the investment technology changes the equilibrium, which, in turn, affects the long-run market configurations. We then proceed with the case of quality externalities, i.e.,

$\kappa_A, \kappa_B > 0$. Here, we show that the presence of a quality externality may make the leading firm to lose market dominance (vis-a-vis its competitor, the lagging firm) in order to increase the industry market share (with respect the outside option). However, in some cases, the presence of the quality externality (induced by a patent released on the part of the leading firm) might also lead the lagging firm to dominate the market if the quality externality is more favorable to the lagging firm.

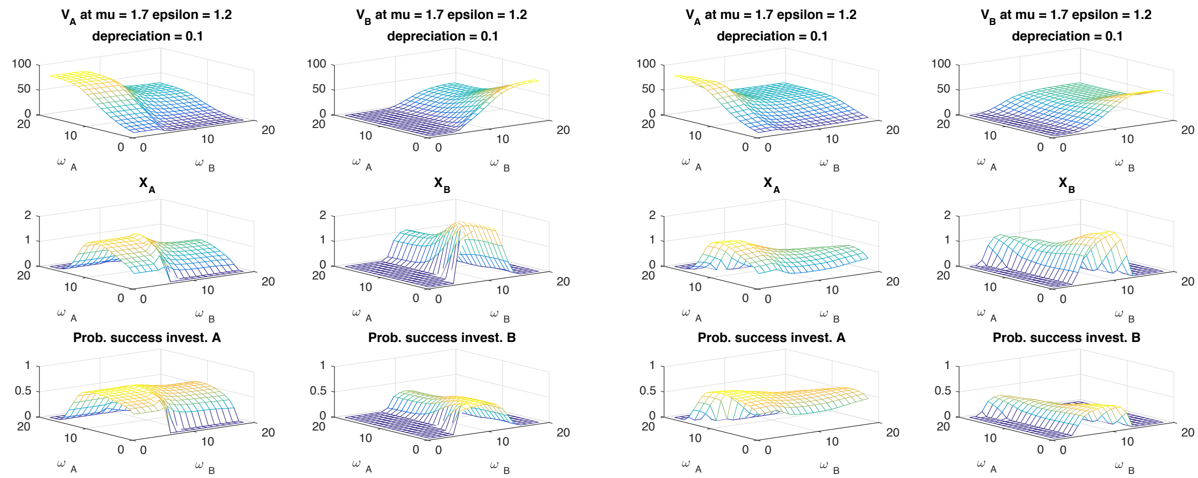
4 Market Structures in the Long-run

4.1 Market Structures in the Absence of Quality Externalities

Suppose that $\kappa_A = \kappa_B = 0$, i.e., there is no quality externality. We concentrate on the case where the probability of an industry-wide negative shock is $\delta = 0.1$ which does not cover the cases of multiplicity of equilibria discussed in Borkovsky et al. (2010). Higher values of δ simply increase the regions of a market collapse in our graphs below. To facilitate the discussion, we parametrize the heterogeneity in α_A and α_B as follows $\alpha_A = \mu$ and $\alpha_B = \mu - \varepsilon$ where $\varepsilon \in [0, \mu]$ measures the heterogeneity in returns to investment between the two firms. Firm A is the leading firm and firm B is the lagging firm.

Differences between α_A and α_B have an effect on the equilibrium investment policy functions and the corresponding probabilities of success. Figure 2 (left panel) provides the converged value and policy functions as well as the corresponding probabilities of success for each firm for the parameter values indicated there.

Figure 2: Value, policy, and probability of success functions



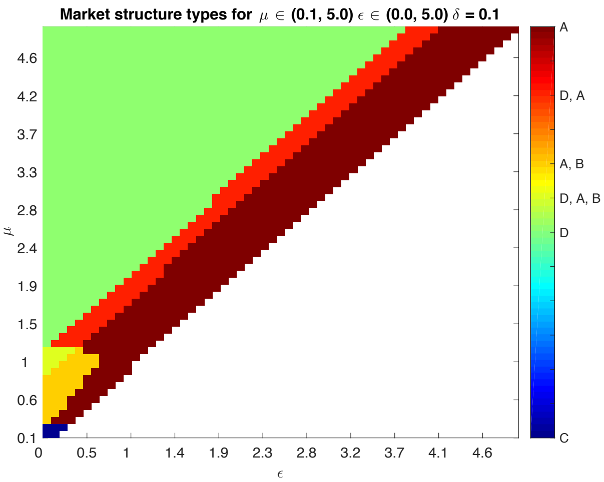
Notes: Left panel: asymmetric R&D capabilities and no externalities. Right panel: Asymmetric R&D capabilities with externalities $\kappa_A = \kappa_B = 0.3$. In both panels $\lambda = 1.7$ and $\delta = 0.1$.

We consider two cases. When the likelihood of success of investment is the same for both firms, the policy and value functions are identical (not shown on the graphs). However, when this likelihood is not the same across the two firms (left panel), the lagging firm (the one with a lower α value, firm B in the graph) invests more in some states to compensate for this lack of likelihood of success. Because of the low probability of success of increasing its product quality and the higher amount of money spent in the investment, firm B receives in the long run a lower stream of cash flows and ends up having lower values for its value function compared to firm A. This is even true when firm B sells a high quality product and firm A is absent (A's quality is equal to 0). The reason for this is that the depreciation effect is strong enough to counteract the possibility of quality improvements, thus leading to low net discounted profits.

Figure 3 provides a general overview of the long-run market configurations for different values of α_A and α_B with $\alpha_A = \mu$ and $\alpha_B = \mu - \varepsilon$: it summarizes all market configurations for different combinations of μ and ε when the rate of depreciation is $\delta = 0.1$. The left panel corresponds to $\lambda = 1.2$ and the right panel to $\lambda = 1.7$. Each point (μ, ε) is associated with one entire probability distribution in the long run such as the one depicted in the right lower panel in Figure 1. Points on the vertical axis represent the cases where both firms are identical ($\alpha_A = \alpha_B$). Any point to the right of the vertical axis represents a case of heterogeneity in which Firm B is the laggard ($\epsilon > 0$). Since below the diagonal the difference

$\mu - \epsilon$ is negative, none of those points are associated with any model specification and they are left in blank. The farther to the right from the vertical axis, the higher the degree of heterogeneity in the likelihood of success of investment. The intensity of color represents each of the market structure types as indicated by the color bar on the right of the graphs.

Figure 3: Market structures when there are no externalities



Notes: The letter A means that firm A dominates the market. The letter D refers to duopoly. The term A, B means that the limiting distribution for quality is bimodal, i.e., either firm may take over as a monopoly. Finally, the letter C indicates that the market collapses. Left panel represents the outcomes $\lambda = 1.2$ and right panel when $\lambda = 1.7$.

As investment becomes more reversible (higher depreciation rate δ) the region for duopoly shrinks from occupying a large portion of the parameter space studied to no presence at all.[16] As the price sensitivity λ increases the outside good market share expands in all cases. This effect dominates when μ is low (below 0.6 on the right panel) and the only market structure prevailing is both firms providing a good with quality of zero (market collapse). For values of μ greater than a certain value (0.6 on the right panel), positive qualities are observed in the long-run but each non-market collapse structure requires a higher value of μ to counteract the higher price sensitivity. The advantage of the leader firm increases in this case.

Finally, we discuss how an increase in heterogeneity (an increase in ε keeping μ constant) leads to changes in the long-run market structures. Figure 3 shows the effect of heterogeneity when a more capable firm (i.e., $\epsilon > 0$), leads the market to change from a duopoly structure to a duopoly and monopoly of the leading firm, and to a monopoly of the leading firm only (for $\mu > 1.5$ on left panel and for $\mu > 2.4$ on the right panel). The effect of heterogeneity is

[16]Further results on the role of δ can be made available upon request.

even stronger when the price sensitivity is higher (right panel). Notice that if the leading firm has a relatively low capacity of transforming investment into succesful increases in quality (low α), then the effect of the heterogeneity is weaker and the probability of observing a monopoly from the laggard coexisting with positive probabilities for duopoly and monopoly from the leader firm is not negligible (μ approximately less than 1.5 on the left panel and less than 2.4 on the right panel). We summarize this insights in the following Observation.

> **Observation 1:** The quality ladder model with heterogeneity in the likelihood of success of investment and quality externalities can exhibit different long-run distributions over market structures depending on parameter values. Those different structures are: market collapse, market dominance by either firm, duopoly, and combinations of these structures.

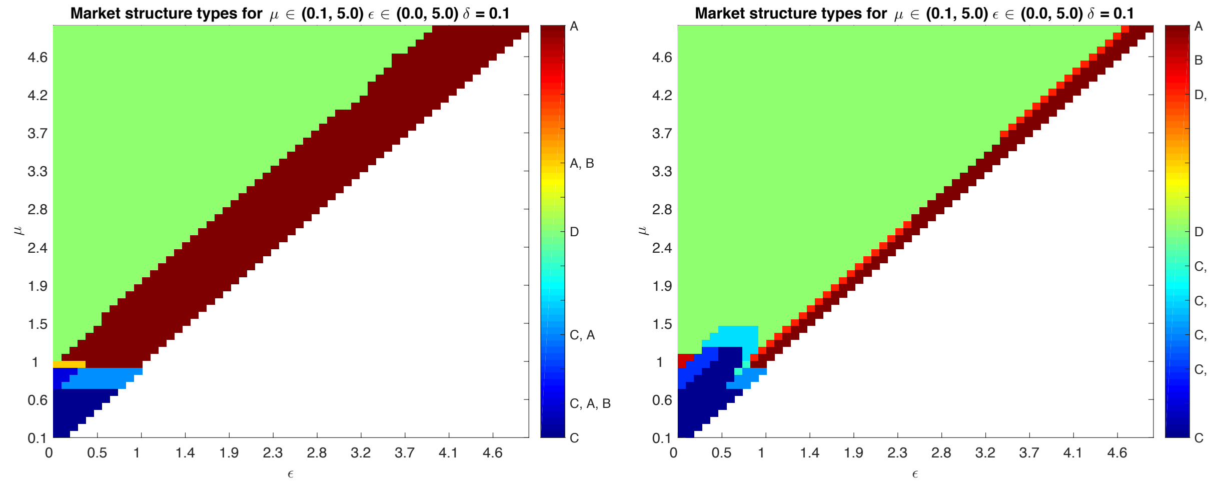
4.2 Market Structures and Quality Externalities

Having discussed the case of heterogeneity in the absence of quality externalities, we now turn to the situation in which there is an externality. This present analysis is motivated by the electric vehicle (EV) market. As mentioned in the introduction, this market has recently witnessed a major change since the leading firm (Tesla Motors) has released most patents, allowing other firms to benefit from Tesla's own improvements in quality.

In our model, the release of the patent has two interpretations. First, consumers' valuation for each good depends on the quality levels of both goods. That is, in the presence of a quality externality, consumers' valuation for good j depends on $\omega_j + \kappa_j \omega_{3-j}$ where $\kappa_j \geq 0$ in the case of a positive externality. Second, the heterogeneity in the return to investment narrows (lower ε). The right panel of Figure 2 shows the policy, value, and probability functions in the case of externalities symmetric externalities. If we compare them against the case of no externalities (left panel) we observe that both firms benefit as indicated by the value function. This occurs because at several points of the state space, the probability of success of investment has increased for both firms, particularly from zero to positive values at some points of the (ω_A, ω_B)-grid.

Figure 4 shows the different combinations of market structures that arise in this model with externalities. There, the left panel shows the results when the externalities are symmetric. The presence of quality externalities shrinks the region in which firm A dominates and makes the duopoly outcome more prevalent. This is due to the fact that the externalities are symmetric, i.e., any firm benefits every time the other firm succeeds.

Figure 4: Market structures when there are externalities



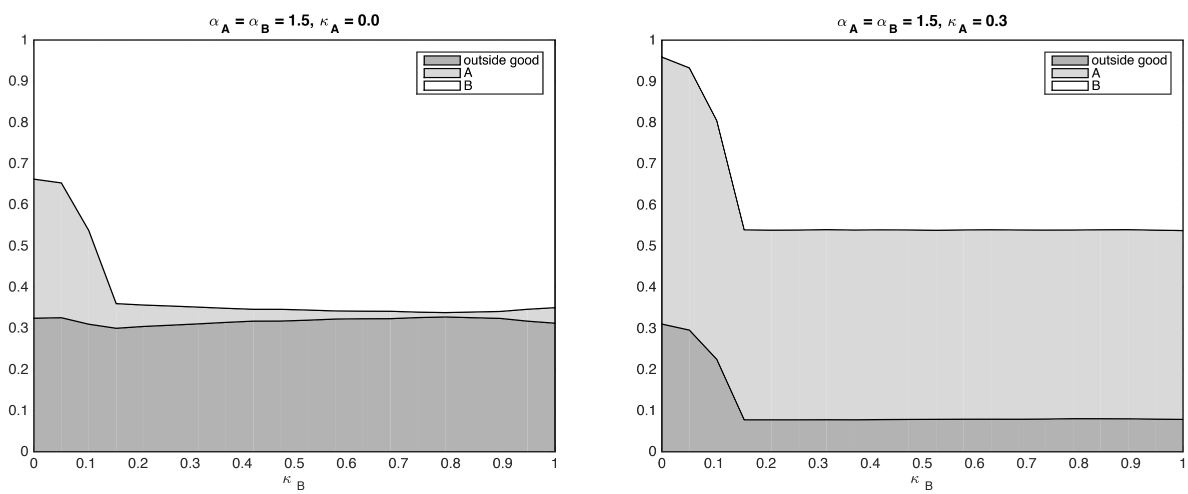
Notes: Left column represents the outcomes from the symmetric externalities case ($\kappa_A = 0.3$ and $\kappa_B = 0.3$). Right column represents outcomes when the externalities are not symmetric ($\kappa_A = 0.3$ and $\kappa_B = 0.7$). Both at $\lambda = 1.7$.

Although the presence of a market externality might be beneficial to the industry, this trade-off can be harmful to the leader if the competitors take advantage of this positive externality (the patent release) to a point where the lagging firm ends up dominating the market. This occurs if the benefit of the lagging firm from the leading firm's quality improvement is strong (i.e., $\kappa_B = 0.7$) whereas the benefit of the leading firm from the lagging firm's quality improvement is weak (i.e., $\kappa_A = 0.3$). This is seen in the right panel of Figure 4. It shows that releasing a patent from a lagging firm might lead to a total loss of market share by A as the lagging firm B takes over the market (near $\mu = 1$ and $\epsilon < 0.1$). We summarize this insight in the following Observation.

> **Observation 2:** Allowing for a quality externality, for instance through the release of a patent, removes market dominance by allowing the lagging firm to benefit from the leading firm's investment. Depending on parameter values, this may lead to dominance by the laggard.

The region in which the laggard ends up dominating the market in the example above might seem too small to be of any major concern. However, the presence of the externality also expands the region in which there is a market collapse, which defeats the purpose of a patent release in an attempt to expand the entire industry's market share. It is common when working with this type of models to report the expected market shares as opposed to the entire distribution over quality: the element-by-element product of the matrix containing the long-run probability distribution and the matrix of market shares from the static game, by doing so, we mask the number of modes in the distribution by providing one single number that may confound different configurations. Figure 5 shows the stacked market shares computed in this manner for different levels of the externality for cases in which the two firms have the same likelihood of success of investment. On the left panel firm A does not receive any externality from B's quality ($\kappa_A = 0$). As κ_B increases, B's market share expands (white region) at the expense of firm A's market share. On the right panel we repeat the same exercise but at a positive level of externality for A, $\kappa_A = 0.3$. Interestingly, in this case B's market share expands and reaches its maximum level even before the point where $\kappa_B = \kappa_A$. This shows that even when masking the potential multiplicity of modes in the long-run distribution by taking expected values of market outcomes, the dominance of the laggard can be observed.

Figure 5: Expected market shares for different levels of the externality



Notes: Each panel shows the stacked expected market shares at each level of the externality for firm B. $\alpha_A = \alpha_B = 1.5$ in both panels. $\kappa_A = 0$ in the first panel and $\kappa_A = 0.3$ in the second. Vertical axis represents market size.

As we concluded in Section 2, the curvature of the investment reaction functions depends on the sign of $\Delta_j - \Psi_j$. For all the cases analyzed here we find that the investments are strategic substitutes.

5 Final Remarks

This paper does not attempt to assess the multiplicity of equilibria that Borkovsky et al. (2010) does. In that case of multiplicity, different policy functions and hence different transient distributions arise from the same parameter values (see for instance Table 7 in that paper). We abstract from such situations by focusing on a region of the parameter space where there has not been documented multiplicity of equilibria. We are agnostic as to whether that phenomenon occurs in the case of externalities but our analysis of the investment reaction functions suggests there is a unique equilibrium. The purpose of our work is to explore the richness of this class of models in terms of long-run distributions. This is known in the literature (for instance Figure 4 in Borkovsky et al. (2012)) but to the best of our knowledge not described in the context of externalities.

As mentioned in the introduction, the estimation of the dynamic quality-ladder model is challenging. One important application is the one in Goettler and Gordon (2011). There, the policy experiments consist of simulating 10,000 industries under some specific counterfactual scenario, each industry is simulated 300 time periods given the initial condition given by the data. Then different outcomes are provided: expected profits, consumer surplus, and investments. It is unlikely but possible that the transient distribution over the quality space exhibits multiple modes, which would indicate the possibility of positive probability of different market structures similar to those in our paper.

Our main application of these insights is to study the effect of a patent release that improves the quality of the competitor's good through a network effect. We model this as a quality externality in the presence of asymmetric returns to investment levels. Such externality is a function of the competitor's product quality and affects the consumer's utility for the other good. We examine the long-run distributions over market structures obtained by simulating the industry using the converged policy and value functions. We show that a single vector of model parameters can generate probability distributions that can lead to positive probabilities for one or more market structures. In particular, we show that it is possible for the laggard to dominate the market, in which case the patent release from the leader should have been avoided. In addition, the potential of multi-modal long-run distributions may have important consequences on the type of market structures that can arise when simulating this type of models.

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Online Appendix (not for publication)

In this appendix, we describe the Levhari-Mirman (LM) (1980) algorithm and the transition matrix.

LM Algorithm

Value Function, Finite Programs. For $j = 1, 2$, consider firm j's maximization problem for a horizon of τ periods, $\tau = 0, 1, ...$ For $j = 1, 2$, given $x_{3-j} \geq 0$, firm j's value function for a τ-period horizon is

$$v_j^{\tau}\left(\omega_j, \omega_{3-j}\right) = \max_{x_j \geq 0} \left\{\Pi_j\left(\omega_j, \omega_{3-j}\right) - x_j + \beta_j \mathbf{E}[v_j^{\tau-1}(\omega_j', \omega_{3-j}')|\omega_j, \omega_{3-j}, x_j, x_{3-j}]\right\} \tag{11}$$

where $\mathbf{E}[\cdot]$ is the expectation operator with respect to $\{\omega_j', \omega_{3-j}'\}$ according to the transition probabilities. The value function for the static game (i.e., when $\tau = 0$) is

$$v_j^{0}\left(\omega_j, \omega_{3-j}\right) = \max_{x_j \geq 0} \left\{\Pi_j\left(\omega_j, \omega_{3-j}\right) - x_j\right\}. \tag{12}$$

Consistent with (11), firm j's value function for the infinite-period horizon is thus

$$v_j^{\infty}\left(\omega_j, \omega_{3-j}\right) = \max_{x_j \geq 0} \left\{\Pi_j\left(\omega_j, \omega_{3-j}\right) - x_j + \beta_j \mathbf{E}[v_j^{\infty}(\omega_j', \omega_{3-j}')|\omega_j, \omega_{3-j}, x_j, x_{3-j}]\right\}. \tag{13}$$

Equilibrium. Next, we define the Markov-perfect equilibrium for a game lasting $T + 1$ periods, i.e., a horizon of T periods, $T = 0, 1, ..., \infty$. The equilibrium consists of the strategies of the two firms for every horizon from the first period (when there are T periods left) to the last period (when there is no horizon).

Definition The tuple $\{X_1^{\tau}(\omega_1, \omega_2), X_2^{\tau}(\omega_2, \omega_1)\}_{\tau=0}^{T}$ is a Markov-perfect Nash equilibrium for a game of T-period horizons if, for all $\{\omega_1, \omega_2\}$,

1. For $\tau = 0$, for $j = 1, 2$, given $X_{3-j}^{0}(\omega_{3-j}, \omega_j)$,

$$X_j^{0}(\omega_{3-j}, \omega_j) = \arg\max_{x_j \geq 0} \left\{\Pi_j\left(\omega_j, \omega_{3-j}\right) - x_j\right\}. \tag{14}$$

2. For $\tau = 1, 2, \ldots, T$, for $j = 1, 2$, given $X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)$ and $\{X^t_1(\omega_1, \omega_2), X^t_2(\omega_2, \omega_1)\}^{\tau-1}_{t=0}$,

$$
\begin{aligned}
& X^{\tau}_j(\omega_{3-j}, \omega_j) \\
& = \arg\max_{x_j \geq 0} \{\Pi_j(\omega_j, \omega_{3-j}) - x_j \\
& + \beta_j \phi_j(x_j) \phi_{3-j}(X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)) \cdot \left(\delta V^{\tau-1}_j(\omega_j, \omega_{3-j}) + (1-\delta) V^{\tau-1}_j(\omega_j + 1, \omega_{3-j} + 1)\right) \\
& + \beta_j \phi_j(x_j)\left(1 - \phi_{3-j}\left(X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)\right)\right) \cdot \left(\delta V^{\tau-1}_j(\omega_j, \omega_{3-j} - 1) + (1-\delta) V^{\tau-1}_j(\omega_j + 1, \omega_{3-j})\right) \\
& + \beta_j (1 - \phi_j(x_j)) \phi_{3-j}\left(X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)\right) \cdot \left(\delta V^{\tau-1}_j(\omega_j - 1, \omega_{3-j}) + (1-\delta) V^{\tau-1}_j(\omega_j, \omega_{3-j} + 1)\right) \\
& + \beta_j (1 - \phi_j(x_j))\left(1 - \phi_{3-j}\left(X^{\tau}_{3-j}(\omega_{3-j}, \omega_j)\right)\right) \cdot \left(\delta V^{\tau-1}_j(\omega_j - 1, \omega_{3-j} - 1) + (1-\delta) V^{\tau-1}_j(\omega_j, \omega_{3-j})\right)\}
\end{aligned} \tag{15}
$$

where, for any $y, z \in \{1, 2, ..., M\}$,

$$
V^{\tau'-1}_j(y, z) = \begin{cases} \Pi_j(y, z) - X^0_j(y, z) & \tau' = 1 \\ \Pi_j(y, z) - X^{\tau'-1}_j(y, z) + \beta_j \cdot \Gamma^{\tau'-2}_j(X^{\tau'-1}_j(y, z), X^{\tau'-1}_{3-j}(z, y)) & \tau' = 2, 3, ..., T \end{cases} \tag{16}
$$

is the value function for a $\tau' - 1$ period horizon for any state vector (y, z) with

$$
\begin{aligned}
& \Gamma^{\tau'-2}_j(X^{\tau'-1}_j(y, z), X^{\tau'-1}_{3-j}(z, y)) \\
& = \phi_j(X^{\tau'-1}_j(y, z)) \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y) \cdot \left(\delta V^{\tau'-2}_j(y, z) + (1-\delta) V^{\tau'-2}_j(y + 1, z + 1)\right) \\
& + \phi_j(X^{\tau'-1}_j(y, z))(1 - \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y))) \cdot \left(\delta V^{\tau'-2}_j(y, z - 1) + (1-\delta) V^{\tau'-2}_j(y + 1, z)\right) \\
& + (1 - \phi_j(X^{\tau'-1}_j(y, z))) \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y)) \cdot \left(\delta V^{\tau'-2}_j(y - 1, z) + (1-\delta) V^{\tau'-2}_j(y, z + 1)\right) \\
& + (1 - \phi_j(X^{\tau'-1}_j(y, z)))(1 - \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y)) \cdot \left(\delta V^{\tau'-2}_j(y - 1, z - 1) + (1-\delta) V^{\tau'-2}_j(y, z)\right)
\end{aligned} \tag{17}
$$

is the expected continuation value function corresponding to the equilibrium for a horizon of $\tau' - 2$ periods.

Condition 1 defines the Nash equilibrium in the static game. Note that in fact, there is no externality since $X^0_{3-j}(\omega_{3-j}, \omega_j)$ has no effect on the zero-period-horizon objective function for firm j. Condition 2 states the equilibrium for every higher horizon of the game. For $\tau = 1, 2, 3, ..., T$, expressions (16) and (17) reflect the recursive nature of the equilibrium in which the equilibrium continuation value function for a $(\tau - 1)$-period horizon depends on the equilibrium strategies for τ'-period horizons, $(\tau - 1) > \tau' \geq 0$.

Proposition states the Markov-perfect Nash equilibrium for each horizon of the game.

Proposition 5.1. *Consider a game of T-period horizons.*

1. *For $\tau = 0$,*
$$\left\{X_1^0\left(\omega_1, \omega_2\right), X_2^0\left(\omega_2, \omega_1\right)\right\} = \{0, 0\}, \tag{18}$$
with the corresponding value function is
$$V_j^0\left(\omega_j, \omega_{3-j}\right) = \Pi_j\left(\omega_j, \omega_{3-j}\right). \tag{19}$$

2. *For $\tau \geq 1$, given $\{V_1^{\tau-1}\left(\omega_1, \omega_2\right), V_2^{\tau-1}(\omega_2, \omega_1)$, $\left\{X_1^\tau\left(\omega_1, \omega_2\right), X_2^\tau\left(\omega_1, \omega_2\right)\right\}$ is defined by*
$$X_1^\tau\left(\omega_1, \omega_2\right) = \max\left\{-\frac{1}{\alpha_1} + \sqrt{\frac{\beta_1}{\alpha_1}}\sqrt{\frac{\alpha_2 X_2^\tau\left(\omega_2, \omega_1\right)\Delta_1^{\tau-1} + \Psi_1^{\tau-1}}{1 + \alpha_2 X_2^\tau\left(\omega_2, \omega_1\right)}}, 0\right\}, \tag{20}$$
$$X_2^\tau\left(\omega_2, \omega_1\right) = \max\left\{-\frac{1}{\alpha_2} + \sqrt{\frac{\beta_2}{\alpha_2}}\sqrt{\frac{\alpha_1 X_1^\tau\left(\omega_1, \omega_2\right)\Delta_2^{\tau-1} + \Psi_2^{\tau-1}}{1 + \alpha_1 X_1^\tau\left(\omega_1, \omega_2\right)}}, 0\right\}, \tag{21}$$
where for $j = 1, 2$,
$$\begin{aligned}
\Delta_j^{\tau-1} &\equiv \delta\left[V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right) - V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j}\right)\right] \\
&\quad + (1-\delta)\left[V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j} + 1\right) - V_j^{\tau-1}\left(\omega_j, \omega_{3-j} + 1\right)\right], \\
\Psi_j^{\tau-1} &\equiv \delta\left[V_j^{\tau-1}\left(\omega_j, \omega_{3-j} - 1\right) - V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j} - 1\right)\right] \\
&\quad + (1-\delta)\left[V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j}\right) - V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right)\right].
\end{aligned}$$

Proof The first-order condition corresponding to (15) is

$$\begin{aligned}
&- d_j + \beta_j \frac{\alpha_j}{\left(1 + \alpha_j x_j\right)^2}\phi_{3-j}(X_{3-j}^\tau(\omega_{3-j}, \omega_j)) \cdot \left(\delta V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right) + (1-\delta)V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j} + 1\right)\right) \\
&+ \beta_j \frac{\alpha_j}{\left(1 + \alpha_j x_j\right)^2}(1 - \phi_{3-j}\left(X_{3-j}^\tau(\omega_{3-j}, \omega_j)\right)) \cdot \left(\delta V_j^{\tau-1}\left(\omega_j, \omega_{3-j} - 1\right) + (1-\delta)V_j^{\tau-1}\left(\omega_j + 1, \omega_{3-j}\right)\right) \\
&- \beta_j \frac{\alpha_j}{\left(1 + \alpha_j x_j\right)^2}\phi_{3-j}\left(X_{3-j}^\tau(\omega_{3-j}, \omega_j)\right) \cdot \left(\delta V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j}\right) + (1-\delta)V_j^{\tau-1}\left(\omega_j, \omega_{3-j} + 1\right)\right) \\
&- \beta_j \frac{\alpha_j}{\left(1 + \alpha_j x_j\right)^2}(1 - \phi_{3-j}\left(X_{3-j}^\tau(\omega_{3-j}, \omega_j)\right)) \cdot \left(\delta V_j^{\tau-1}\left(\omega_j - 1, \omega_{3-j} - 1\right) + (1-\delta)V_j^{\tau-1}\left(\omega_j, \omega_{3-j}\right)\right) \\
&= 0
\end{aligned}$$

which yields (9) and thus (21), as long as the second-order condition is satisfied, i.e., for

$j, 3-j = 1, 2, j \neq 3-j,$

$$
\begin{aligned}
&-\beta_j \frac{2\alpha_j^2}{(1+\alpha_j x_j)^3} \frac{\alpha_{3-j} x_{3-j}}{1+\alpha_{3-j} x_{3-j}} \cdot \left(\delta V_j^{\tau-1}(\omega_j, \omega_{3-j}) + (1-\delta) V_j^{\tau-1}(\omega_j+1, \omega_{3-j}+1)\right) \\
&-\beta_j \frac{2\alpha_j^2}{(1+\alpha_j x_j)^3} \frac{1}{1+\alpha_{3-j} x_{3-j}} \cdot \left(\delta V_j^{\tau-1}(\omega_j, \omega_{3-j}-1) + (1-\delta) V_j^{\tau-1}(\omega_j+1, \omega_{3-j})\right) \\
&+\beta_j \frac{2\alpha_j^2}{(1+\alpha_j x_j)^3} \frac{\alpha_{3-j} x_{3-j}}{1+\alpha_{3-j} x_{3-j}} \cdot \left(\delta V_j^{\tau-1}(\omega_j-1, \omega_{3-j}) + (1-\delta) V_j^{\tau-1}(\omega_j, \omega_{3-j}+1)\right) \\
&+\beta_j \frac{2\alpha_j^2}{(1+\alpha_j x_j)^3} \frac{1}{1+\alpha_{3-j} x_{3-j}} \cdot \left(\delta V_j^{\tau-1}(\omega_j-1, \omega_{3-j}-1) + (1-\delta) V_j^{\tau-1}(\omega_j, \omega_{3-j})\right) < 0
\end{aligned}
$$

Algorithm. Having described the model and defined the equilibrium. We now proceed with the characterization of the MPE. Here, we solve the equilibrium recursively as in Levhari and Mirman (1980). Consider first the static game of investment, i.e., $\tau = 0$. Then, there is no externality, and no firm has an incentive to invest, i.e., the Markov-perfect equilibrium for a game of 0-periods horizon is simply

$$\left\{X_1^1(\omega_1, \omega_2), X_2^1(\omega_1, \omega_2)\right\} = \{0, 0\},$$

with the corresponding value function

$$V_j^0(\omega_j, \omega_{3-j}) = \Pi_j(\omega_j, \omega_{3-j}).$$

Hence, there is a unique equilibrium for the no-horizon game in which the firms do not invest and the value function is equal to the profit function corresponding to the Bertrand game.

Consistent with the solution of the equilibrium, we characterize the equilibrium for each horizon. Each iteration is an horizon with the caveat that at each iteration, the solution to the reaction function is a Markov-perfect Nash equilibrium (and not an approximation). Hence, wherever we stop, we have an equilibrium. The question remains whether we converge to the stationary Markov-perfect Nash equilibrium (in infinite horizons).

1. For $\tau = 0$,

   $$\left\{X_1^0(\omega_1, \omega_2), X_2^0(\omega_2, \omega_1)\right\} = \{0, 0\},$$

   with the corresponding value function is

   $$V_j^0(\omega_j, \omega_{3-j}) = \Pi_j(\omega_j, \omega_{3-j}).$$

2. For $\tau \geq 1$, given $\{V_1^{\tau-1}(\omega_1, \omega_2), V_2^{\tau-1}(\omega_2, \omega_1)\}$, firm j's reaction functions are given by (21) and the solution to this system of equations is the equilibrium at this iteration.

Transition Probability Matrix

Using the converged policy functions, for $j = 1, 2$,

$$\omega_j' | \omega_j = \min\{\max\{\omega_j + \tau_j + \eta, 1\}, M\}$$

where $\tau_j \in \{1, 0\}$ such that $\Pr[\tau_j = 1] = \phi_j(\omega_1, \omega_2) = \frac{\alpha_j X_j(\omega_j, \omega_{3-j})}{1+\alpha_j X_j(\omega_j, \omega_{3-j})}$ and $\eta \in \{-1, 0\}$ such that $\Pr[\eta = -1] = \delta$.

We want to calculate all transition probabilities such as $\Pr[(\omega_1', \omega_2') \,|\, (\omega_1, \omega_2)]$. We consider each case separately.

1. Suppose that (ω_1, ω_2) is such that $\omega_1, \omega_2 \notin \{0, M\}$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}
\Pr[(\omega_1, \omega_2) \,|\, (\omega_1, \omega_2)] &= \delta \phi_1(\omega_1, \omega_2) \phi_2(\omega_2, \omega_1) \\
&\quad + (1-\delta)(1 - \phi_1(\omega_1, \omega_2))(1 - \phi_2(\omega_2, \omega_1)), \\
\Pr[(\omega_1 + 1, \omega_2) \,|\, (\omega_1, \omega_2)] &= (1-\delta)\phi_1(\omega_1, \omega_2)(1 - \phi_2(\omega_2, \omega_1)), \\
\Pr[(\omega_1 - 1, \omega_2) \,|\, (\omega_1, \omega_2)] &= \delta(1 - \phi_1(\omega_1, \omega_2)) \phi_2(\omega_2, \omega_1), \\
\Pr[(\omega_1, \omega_2 - 1) \,|\, (\omega_1, \omega_2)] &= \delta \phi_1(\omega_1, \omega_2)(1 - \phi_2(\omega_2, \omega_1)), \\
\Pr[(\omega_1 - 1, \omega_2 - 1) \,|\, (\omega_1, \omega_2)] &= \delta(1 - \phi_1(\omega_1, \omega_2))(1 - \phi_2(\omega_2, \omega_1)), \\
\Pr[(\omega_1, \omega_2 + 1) \,|\, (\omega_1, \omega_2)] &= (1-\delta)(1 - \phi_1(\omega_1, \omega_2)) \phi_2(\omega_2, \omega_1), \\
\Pr[(\omega_1 + 1, \omega_2 + 1) \,|\, (\omega_1, \omega_2)] &= (1-\delta) \phi_1(\omega_1, \omega_2) \phi_2(\omega_2, \omega_1).
\end{aligned}$$

2. Suppose that $(\omega_1, \omega_2) = (0, 0)$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}
\Pr[(0,0) \,|(0,0)] &= 1 - (1-\delta)(\phi_1(0,0) + \phi_2(0,0) - \phi_1(0,0)\phi_2(0,0)), \\
\Pr[(1,0) \,|(0,0)] &= (1-\delta)\phi_1(0,0)(1 - \phi_2(0,0)), \\
\Pr[(0,1) \,|(0,0)] &= (1-\delta)(1 - \phi_1(0,0))\phi_2(0,0), \\
\Pr[(1,1) \,|(0,0)] &= (1-\delta)\phi_1(0,0)\phi_2(0,0).
\end{aligned}$$

3. Suppose that $(\omega_1, \omega_2) = (M, M)$. Given (ω_1, ω_2), there are $(M+1)^2$ conditional prob-

abilities to calculate. All of them are zero except

$$
\begin{aligned}
\Pr[(M,M)\,|\,(M,M)] &= 1-\delta(1-\phi_1(M,M)\,\phi_2(M,M)),\\
\Pr[(M-1,M)\,|\,(M,M)] &= \delta\,(1-\phi_1(M,M))\,\phi_2(M,M)\,,\\
\Pr[(M,M-1)\,|\,(M,M)] &= \delta\phi_1(M,M)\,(1-\phi_2(M,M))\,,\\
\Pr[(M-1,M-1)\,|\,(M,M)] &= \delta\,(1-\phi_1(M,M))\,(1-\phi_2(M,M))\,.
\end{aligned}
$$

4. Suppose that $(\omega_1,\omega_2)=(0,M)$. Given (ω_1,ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$
\begin{aligned}
\Pr[(0,M)\,|\,(0,M)] &= 1-(1-\delta)\phi_1(0,M)-\delta\,(1-\phi_2(0,M))\,,\\
\Pr[(1,M)\,|\,(0,M)] &= (1-\delta)\phi_1(0,M)\,,\\
\Pr[(0,M-1)\,|\,(0,M)] &= \delta\,(1-\phi_2(0,M))\,.
\end{aligned}
$$

5. Suppose that $(\omega_1,\omega_2)=(M,0)$. Given (ω_1,ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$
\begin{aligned}
\Pr[(M,0)\,|\,(M,0)] &= 1-(1-\delta)\phi_2(0,M)-\delta\,(1-\phi_1(M,0))\,,\\
\Pr[(M,1)\,|\,(M,0)] &= (1-\delta)\phi_2(0,M)\,,\\
\Pr[(M-1,0)\,|\,(M,0)] &= \delta\,(1-\phi_1(M,0))\,.
\end{aligned}
$$

6. Suppose that (ω_1,ω_2) is such that $\omega_1=0$ and $\omega_2\notin\{0,M\}$. Given (ω_1,ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$
\begin{aligned}
\Pr[(0,\omega_2)\,|\,(0,\omega_2)] &= \delta\phi_2(\omega_2,0)\\
&\quad+(1-\delta)\,(1-\phi_1(0,\omega_2))\,(1-\phi_2(\omega_2,0))\,,\\
\Pr[(1,\omega_2)\,|\,(0,\omega_2)] &= (1-\delta)\phi_1(0,\omega_2)\,(1-\phi_2(\omega_2,0))\,,\\
\Pr[(0,\omega_2-1)\,|\,(0,\omega_2)] &= \delta\,(1-\phi_2(\omega_2,0))\,,\\
\Pr[(0,\omega_2+1)\,|\,(0,\omega_2)] &= (1-\delta)\,(1-\phi_1(0,\omega_2))\,\phi_2(\omega_2,0)\,,\\
\Pr[(1,\omega_2+1)\,|\,(0,\omega_2)] &= (1-\delta)\,\phi_1(0,\omega_2)\,\phi_2(\omega_2,0)\,.
\end{aligned}
$$

7. Suppose that (ω_1,ω_2) is such that $\omega_1\notin\{0,M\}$ and $\omega_2=0$. Given (ω_1,ω_2), there are

$(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}
\Pr[(\omega_1,0)\,|\,(\omega_1,0)] &= \delta\phi_1\left(\omega_1,0\right) \\
&\quad + (1-\delta)\left(1-\phi_2\left(0,\omega_1\right)\right)\left(1-\phi_1\left(\omega_1,0\right)\right), \\
\Pr[(\omega_1,1)\,|\,(\omega_1,0)] &= (1-\delta)\phi_2\left(0,\omega_1\right)\left(1-\phi_1\left(\omega_1,0\right)\right), \\
\Pr[(\omega_1-1,0)\,|\,(\omega_1,0)] &= \delta\left(1-\phi_1\left(\omega_1,0\right)\right), \\
\Pr[(\omega_1+1,0)\,|\,(\omega_1,0)] &= (1-\delta)\left(1-\phi_2\left(0,\omega_1\right)\right)\phi_1\left(\omega_1,0\right), \\
\Pr[(\omega_1+1,1)\,|\,(\omega_1,0)] &= (1-\delta)\,\phi_2\left(0,\omega_1\right)\phi_1\left(\omega_1,0\right).
\end{aligned}$$

8. Suppose that (ω_1,ω_2) is such that $\omega_1 = M$ and $\omega_2 \notin \{0, M\}$. Given (ω_1,ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}
\Pr[(M,\omega_2)\,|\,(M,\omega_2)] &= \delta\phi_1\left(M,\omega_2\right)\phi_2\left(\omega_2,M\right) \\
&\quad + (1-\delta)(1-\phi_2\left(\omega_2,M\right)) \\
\Pr[(M-1,\omega_2)\,|\,(M,\omega_2)] &= \delta\left(1-\phi_1\left(M,\omega_2\right)\right)\phi_2\left(\omega_2,M\right) \\
\Pr[(M,\omega_2-1)\,|\,(M,\omega_2)] &= \delta\phi_1\left(M,\omega_2\right)\left(1-\phi_2\left(\omega_2,M\right)\right) \\
\Pr[(M-1,\omega_2-1)\,|\,(M,\omega_2)] &= \delta\left(1-\phi_1\left(M,\omega_2\right)\right)\left(1-\phi_2\left(\omega_2,M\right)\right) \\
\Pr[(M,\omega_2+1)\,|\,(M,\omega_2)] &= (1-\delta)\phi_2\left(\omega_2,M\right)
\end{aligned}$$

9. Suppose that (ω_1,ω_2) is such that $\omega_1 \notin \{0, M\}$ and $\omega_2 = M$. Given (ω_1,ω_2), there are $(M+1)^2$ conditional probabilities to calculate. All of them are zero except

$$\begin{aligned}
\Pr[(M,\omega_1)\,|\,(\omega_1,M)] &= \delta\phi_2\left(M,\omega_1\right)\phi_1\left(\omega_1,M\right) \\
&\quad + (1-\delta)(1-\phi_1\left(\omega_1,M\right)) \\
\Pr[(\omega_1,M-1)\,|\,(\omega_1,M)] &= \delta\left(1-\phi_2\left(M,\omega_1\right)\right)\phi_1\left(\omega_1,M\right) \\
\Pr[(\omega_1-1,M)\,|\,(\omega_1,M)] &= \delta\phi_2\left(M,\omega_1\right)\left(1-\phi_1\left(\omega_1,M\right)\right) \\
\Pr[(\omega_1-1,M-1)\,|\,(\omega_1,M)] &= \delta\left(1-\phi_2\left(M,\omega_1\right)\right)\left(1-\phi_1\left(\omega_1,M\right)\right) \\
\Pr[(\omega_1+1,M)\,|\,(\omega_1,M)] &= (1-\delta)\phi_1\left(\omega_1,M\right).
\end{aligned}$$