2007s-10

Extracting Several Resource Deposits of Unknown Size: Optimal Order

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Série Scientifique
Scientific Series

Montréal
Avril 2007

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ISSN 1198-8177

Partenaire financier

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Extracting Several Resource Deposits of Unknown Size: Optimal Order

Murray C Kemp[*], *Ngo Van Long*[†]

Résumé / *Abstract*

Les compagnies pétrolières révisent souvent les chiffres de leurs réserves, ce qui indique que l'incertitude concernant les stocks est prévalente. Nous considérons le cas où l'extraction donne des informations sur la taille des réserves. Nous prouvons que l'ordre optimal d'exploitation des stocks dépend des propriétés du processus d'extraction concernant la révélation d'information et des coûts. La différence des coûts, qui est une considération importante dans Solow and Wan (1976), doit être balancée contre la valeur informative des réserves. Notre modèle fournit une explication du fait que les réserves plus coûteuses sont parfois exploitées avant l'épuisement des réserves moins coûteuses.

Mots clés : ordre d'extraction, valeur de l'information, incertitude

Oil companies often announce revised estimates of their reserves. This indicates that stock uncertainty is a prevalent feature of natural resource industries. In this paper we consider the multi-deposit case where resource extraction produces information about the size of reserves. We show that the optimal order of extracting resource deposits depends both on the informational characteristics of the extraction process and on the extraction costs. Differences in extraction costs, a key consideration highlighted in Solow and Wan (1976), must be balanced against the relative value of information generated by the extraction of various deposits. Our model supplies an explanation of why high cost deposits are sometimes extracted when lower cost deposits have not been exhausted.

Keywords: *order of extraction, value of information, uncertainty*

Codes JEL : Q30

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1. Introduction

The problem of determining the optimal extraction of a resource deposit of unknown size was first posed by Kemp (1976), and later discussed in Kemp (1977), Kemp and Long (1980a, 1985), Gilbert (1979), Loury (1978), and Kumar (2002, 2005), among others. Another question in the theory of exhaustible resources is the optimal order of extraction of deposits of known sizes, with different extraction costs. This question was first raised by Herfindahl (1967)[1], and subsequently taken up by several authors, including Solow and Wan (1976), Kemp and Long (1980b), Amigues, Gaudet, Favard and Moreaux (1998), Amigues, Longand Moreaux (2006).

The two problems share a common concern: what is the correct time path of the charge to users of extracted resources? In the simplest resource extraction problem, considered by Hotelling (1931), where there is complete certainty, and only one deposit, the shadow price of the stock must rise at a rate equal to the rate of interest[2]. This implies that the net price (i.e., consumer's price net of marginal extraction cost) must rise at the rate of interest. This is known as Hotelling's Rule[3]. Herfindahl (1967) considers the case of several known deposits with different extraction costs, and shows in a partial equilibrium setting that a lower cost deposit should be exhausted before extraction of the higher cost deposit begins. This implies that even though the shadow price of each deposit rises at the rate of interest, the net price (consumer's price minus marginal extraction cost) does not[4]: while the time path of users' price is continuous, the net price jumps down

[1] For a comparative static analysis of the Herfindahl model, see Hartwick (1978).

[2] For an earlier theoretical treatment of the resource extracting firm, see Gray (1916).

[3] Of course if extraction cost is stock-dependent, Hotelling's Rule must be modified. See Levhari and Leviatan (1977), Kemp and Long (1980c).

[4] For a simple diagramatic exposition, see Dasgupta and Heal (1979, Diagram 6.4, p. 173).

(by an amount equal to the difference in marginal extraction costs) at transition points. Because of these jumps, it follows that net price, on average, rises at a rate lower than the rate of interest. Solow and Wan (1976) confirm Herfindahl's result in a general equilibrium setting where capital accumulation takes place and hence the interest rate is endogenous. Furthermore, they show that, in the case of a continuum of deposits (so that every point is a "transition point"), the net price must be larger than the shadow price of the aggregate resource stock by a factor $1+q$, where q is the shadow *surcharge* (over and above the shadow price) for the use of the resource[5].

Kemp (1976), investigating the optimal extraction of a resource stock of unknown size, shows that net price is possibly non-monotone, and hence generically not rising at the rate of interest, but for a different reason: the concave utility function of the planner implies a precautionary motive in the face of stock size uncertainty. If one does not know how much one has, one must proceed with caution. How much caution is optimal at any given time depends on the "hazard rate" at that time. In general the hazard rate is not a constant[6]. In fact, the time path of the hazard rate can be influenced by the choice of the planned extraction path. As extraction proceeds, news arrives continuously: e.g., one learns what is the probability that the next million barrells of oil is available.

The present paper raises the following question: what is the optimal order of exploitation when several deposits are of unknown sizes? Suppose there are two resource deposits, each of unknown size. How should one exploit them? In our search for an answer to this question, we find it convenient to begin with a simple model of optimal extraction of a single two-layered deposit of unknown size. The information about the size of the second layer arrives as soon as the first layer has been exhausted. How fast should one exhaust the first layer in view

[5] Solow and Wan (1976), p. 365.

[6] The exceptional case of a constant hazard rate (i.e., the distribution is exponential) gives a simple extraction rule; this case is exploited by Loury (1978) and Robson (1979).

of one's ability to influence the information-arrival date? What is the relationship between the terminal rate of extraction of the first layer and the initial rate of extraction of the second layer? This section is followed by a generalisation to the case of two deposits of unknown size, each composed of two layers. Good news or bad news about the second layer arrives as soon as the top layer is exhausted. Finally, we consider a further generalisation: the case where, for each mine, some learning about its second layer takes place during the extraction process of its first layer that permits revision of probabilities. One thus receives "little good news" or "little bad news" some time before the arrival of the big news about the second layers. We characterize the optimal extraction order, and in some specific cases, give formulas for computing certainty equivalents and for deciding whether one sequence dominates another.

The implication of our finding for the net price of an exhaustible resource is that it should reflect the informational value of extraction. The shadow price of a homogeneous first layer may not rise at the rate of interest: there is a jump in the shadow price when a little bad news or a little good news arrive. It may be optimal to extract the top half of a first layer more quickly to hasten the arrival of information. This entails a higher initial extraction path, hence a fall in consumer's price, reflecting an "informational premium" (in contrast to the "surcharge" derived by Solow and Wan, 1976, which is derived under conditions of certainty and which reflects the transition to a higher cost deposit). In our model, in order to focus on the informational value about reserve sizes, we abstract from differences in extraction costs. In a more general model, both our informational premium and the Solow-Wan extraction cost surcharge would be combined to obtain the correct pricing.

Before proceding to our formal model, we note that, prior to the present study, there has been some discussion of the issue of sequential extraction under stock uncertainty, but results have been sparse. Kemp (1977) formulated the problem of extracting a sequence of deposits, including the case where some deposits become available only in the future, with unknown delivery date. Robson (1979) found that if the distribution of deposit size is exponential and is identical for both deposits, then the order of extraction is a matter of indifference, while if the exponential distribution of deposit 1 has a lower mean than that of deposit 2 then one should exhaust deposit 1 first. Hartwick (1983) characterized the jump in the price path[7] when extraction of a new deposit begins.

Our paper can be placed in a more general context that encompasses several inter-related issues: the value of information, the timing of the resolution of uncertainty, and the choice of which uncertainty to resolve first. Following the seminal work of Blackwell (1951), economists have developed models about optimal learning. Long and Manning (1972) and Kihlstrom (1974) showed how consumers can optimally learn about product quality. Long (1976) explored the implication of Bayesian learning in a model of foreign investment. Grossman, Kihlstrom and Mirman (1977) addressed the issue of production of information and learning by doing. Gittins (1979) demonstrated that the solution to a class of learning problems consists of choosing at each stage the action with the largest "dynamic allocation index"[8]. Epstein (1980) assumed away optimal learning, and focused instead on the effect of the exogenous resolution of uncertainty on decision. Hartwick and Yeung (1985, 1988, 1989) found conditions under which a value function is convex in a random variable (such as future prices, interest rates, or production costs). In our paper, the decision maker can choose (i) the time of arrival of information, (ii) which uncertainty to be resolved first, (iii) whether to give priority to obtaining full information on a mine, or partial information about two mines. These choices involve costs: early exhaustion of a layer to obtain information goes against the normal desire of consumption smoothing. The

[7] Jumps in price paths were also discussed in Dasgupta and Heal (1979, p. 428-433) in the context of exploration for new reserves, and in Hartwick, Kemp and Long (1986) in the context of set-up costs.

[8] The dynamic allocation index has since become known as the Gittins index. See for example Brezzi and Lai (2000).

optimal choice must strike the right balance.

Finally, we would like to emphasize that the kind of uncertainty we deal with in this paper is not influenced by future events. The size of a deposit is unknown to the decision maker, but has been determined by geological history. We are not dealing with "event uncertainty": some future events may have impacts on the size of a recoverable stock of natural resource. Tsur and Zemel (2004) mentioned pollution-induced events (Cropper, 1976, Tsur and Zemel, 1996), forest fires (Reeds, 1984), sea-water intrusions (Tsur and Zemel, 1995), political events (Long, 1975, Tsur and Zemel, 1998). Due to limitation of space, we refrain from discussing the implications of event uncertainty.

2. One deposit of unknown size

Assume we have a mineral deposit with two layers, and one must exhaust the first layer before reaching the second one. The size of the first layer is A, a known positive number. The size of the second layer is a random variable X which can be zero or x, a known positive number. (Here we use the convention that the capital letter X denotes a random variable, while the lower case x is an actual value.) The subjective probability that $X = x$ is $p > 0$ and the subjective probability that $X = 0$ is $1 - p > 0$. **We assume that the subjective uncertainty is resolved as soon as the first layer, A, is exhausted.** The resource is not storable after extraction. If the decision wants to advance the date at which information becomes available, he must extract at a faster rate. How fast should the planner exhaust A?

(Think of being invited to a dinner under imperfect information: you see the first course, but do not know if a second course will be offered after completing the first course.)

Let $q(t)$ be the rate of extraction at time t. The marginal cost of extraction is $c \geq 0$. The utility function is strictly concave and increasing:

$$U = U(q) \text{ with } U(0) = 0,\ U' > 0 \text{ and } U'' < 0.$$

Whenever explicit solutions are required, we will use the following functional form.

Special functional form

Constant elasticity of marginal utility (CEMU)

$$U(q) = \frac{q^{1-\beta}}{1-\beta}$$

We assume $\beta \neq 1$. The CEMU function is strictly concave and increasing for all $\beta > 0$. However, in what follows, we do not consider the case $\beta > 1$, because $\beta > 1$ implies $U(0) = -\infty$, and this would cause awkward problems concerning existence of optimal paths when the size of the cake is unknown.

After the exhaustion of A (at some time T) the consumer receives either "good news", i.e. $X = x$, or "bad news", i.e., $X = 0$. In the case of bad news, her discounted utility stream (from time T^+ to time infinity) is zero. In the case of good news, it is $e^{-rT^+}V(x)$ where $V(x)$ is the solution to the following problem.

Problem S (Optimal consumption of the second layer)

$$V(x) \equiv \max_{q(.)} \int_{T^+}^{\infty} \left[U(q(t)) - cq(t)\right] e^{-r(t-T^+)} dt$$

subject to

$$\int_{T^+}^{\infty} q(t)dt = x$$

The following lemmas will be useful.

Lemma 1 *At any time $t \geq T^+$, the current-value Hamiltonian of the resource extraction problem (Problem S), when evaluated at the optimal choice, is equal to the consumer surplus, $U(q(t)) - U'(q(t)).q(t)$.*

Proof The present-value Hamiltonian at t is

$$H(t) = \left[U(q(t)) - cq(t)\right] e^{-r(t-T^+)} - \lambda q(t)$$

where $\lambda \geq 0$ is the constant present-value shadow price of the stock x. The current value Hamiltonian is

$$H^C(t) = U(q(t)) - cq(t) - \lambda e^{r(t-T^+)} q(t)$$

We have the necessary condition

$$\left[U'(q(t)) - c\right] e^{-r(t-T^+)} = \lambda$$

Substituting this condition into the current-value Hamiltonian, we obtain

$$H^C(t) = U(q(t)) - U'(q(t)).q(t)\text{..........}\blacksquare$$

For convenience, we denote the consumer surplus function by $CS(q)$:

$$CS(q) \equiv U(q) - U'(q).q$$

Lemma 2 *The value function $V(x)$ is equal to the present value of a (fictitious) perpetual stream of consumer surplus $CS(q(T^+))$, that is,*

$$V(x) = \frac{CS(q(T^+))}{r} = \frac{U(q(T^+)) - U'(q(T^+)).q(T^+)}{r}$$

Proof Use the Hamilton-Jacobi-Bellman relationship $H^C(T^+) = rV(x(T^+))$ $\blacksquare$

Remark If follows from Lemma 2 that the present value of the optimal declining stream of utility (net of extraction costs) is equal to the present value of a (fictitious) perpetual stream of constant consumer surplus $CS(q(T^+))$

$$\int_{T^+}^{\infty} \left[U(q(t)) - cq(t)\right] e^{-r(t-T^+)} dt = \frac{CS(q(T^+))}{r}$$

Before receiving the news, the expected utility from layer X is

$$EV(X) \equiv pV(x) + (1-p)V(0)$$

We now characterize the optimal plan to extract from layer A. This plan consists of an optimal terminal time T, an optimal terminal extraction rate from layer A, denoted by q_A^{term}, and an optimal time path $q(.)$ which is easily pinned down, once we have determined q_A^{term}. The optimization problem is as follows.

Problem W Find T, q_A^{term}, and $q_A(t)$ for $t \in [0, T]$ that solve

$$W(A) = \max_{T,q,q_A^{term}} \int_0^T e^{-rt} \left[U(q_A(t)) - cq_A(t) \right] dt + e^{-rT} EV(X)$$

subject to

$$\int_0^T q_A(t)dt = A$$

where $q_A(t) \geq 0$ and $q_A(T) = q_A^{term}$.

It turns out that there is a simple condition that characterizes q_A^{term}, and it does not depend on the size of A. This condition is given by Proposition 1.

Proposition 1 *The optimal terminal extraction rate from layer A, denoted by q_A^{term}, must satisfy the condition that the consumer surplus at q_A^{term} is equated the product of the interest rate and the expected utility $EV(X)$:*

$$CS(q_A^{term}) = rEV(X) \tag{1}$$

It follows that

$$q_A^{term} = CS^{-1} \left(rEV(X) \right) \tag{2}$$

where $CS^{-1}(.)$ is the inverse of the consumer surplus function.

Proof The present-value Hamiltonian for Problem W is

$$H_A(t) = e^{-rt} \left[U(q_A(t)) - cq_A(t) \right] - \lambda_A q_A(t)$$

where λ_A is the constant present-value shadow price of layer A. The necessary conditions include:

(i) The Hotelling Rule

$$\left[U'(q_A(t)) - c \right] e^{-rt} = \left[U'(q_A(T)) - c \right] e^{-rT} = \lambda_A$$

(ii) The transversality condition[9]

$$H_A(T) - re^{-rT} EV(X) = 0$$

[9] See for example Leonard and Long (1992, Chapter 6).

Using the Hotelling Rule,

$$H_A(T) = e^{-rT}\left[U(q_A(T)) - U'(q_A(T)).q_A(T)\right] = e^{-rT} CS(q_A(T))$$

Combining this result with the transversality condition, we obtain $CS(q_A^{term}) = rEV(X)$ ■

Example 1 Assume the utility function CEMU, and zero extraction cost. It is easy to show that

$$V(x) = \left(\frac{r}{\beta}\right)^{-\beta}\left(\frac{1}{1-\beta}\right) x^{1-\beta}$$

and the initial rate of extraction (from the second layer) at time T^+ , denoted by q_X^{ini} , is given by

$$(q_X^{ini})^{-\beta} = U'(q_X^{ini}) = V'(x) = x^{-\beta}\left(\frac{r}{\beta}\right)^{-\beta} = \left(\frac{xr}{\beta}\right)^{-\beta}$$

i.e.,

$$q_X^{ini} = \frac{xr}{\beta}$$

It follows that

$$EV(X) = p\left(\frac{r}{\beta}\right)^{-\beta}\left(\frac{1}{1-\beta}\right) x^{1-\beta} + (1-p).0 \tag{3}$$

Under CEMU, the consumer surplus function is

$$CS(q) = \frac{\beta}{1-\beta} q^{1-\beta}$$

Applying Proposition 1, we get

$$CS(q_A^{term}) = \frac{\beta}{1-\beta}(q_A^{term})^{1-\beta} = rEV(X) = rp\left(\frac{r}{\beta}\right)^{-\beta}\left(\frac{x^{1-\beta}}{1-\beta}\right) \tag{4}$$

Thus

$$\frac{EV(X)}{(q_A^{term})^{1-\beta}} = \frac{\beta}{r(1-\beta)} \tag{5}$$

and

$$q_A^{term} = CS^{-1}(rEV(X)) = \left[\left(\frac{1-\beta}{\beta}\right)(rp)\left(\frac{r}{\beta}\right)^{-\beta}\left(\frac{x^{1-\beta}}{1-\beta}\right)\right]^{\frac{1}{1-\beta}}$$

$$q_A^{term} = \left(\frac{r}{\beta}\right) xp^{\frac{1}{1-\beta}}$$

Note that in this case, q_A^{term} is homogeneous of degree one in x. (This is due to the CEMU utility function, and zero extraction cost.)

Remark a It follows that $q_A^{term} = p^{\frac{1}{1-\beta}}.q_X^{ini}$; hence $q_X^{ini} > q_A^{term}$, i.e., the rate of extraction jumps up when the good news arrives.

Remark b As is clear from equation (3), under CEMU and zero extraction costs, the expected utility of the prospect $(x, 0; p, 1-p)$ is equal to the expected utility of the prospect $(y, 0; 1, 0)$ if and only if $y = xp^{\frac{1}{1-\beta}}$. In other words, as far as expected utility is concerned, the **certainty-equivalent** of $(x, 0; p, 1-p)$ is a stock y such that $y = xp^{\frac{1}{1-\beta}}$.

Corollary 1

(i)The optimal extraction rate $q_A(t)$ satisfies

$$q_A(t) = U'^{-1}\left[e^{-r(T-t)}(U'(q_A^{term}) - c) + c\right] \equiv Q_A(T-t, q_A^{term})$$

where, for given T, $Q_A(T-t, q_A^{term})$ is an increasing function of q_A^{term}.

(ii)The optimal terminal time for layer A satisfies

$$\int_0^T Q_A(T-t, q_A^{term})dt = A$$

Therefore

$$T = T(A, q_A^{term}) = T(A, CS^{-1}(rEV(X)))$$

where $T(.)$ is an increasing function of A and a decreasing function of q_A^{term}.

Proof: omitted.

Example 1 (continued) Under CEMU and zero extraction costs,

$$[q_A(t)]^{-\beta} e^{-rt} = \left[q_A^{term}\right]^{-\beta} e^{-rT}$$

Hence

$$q_A(t) = q_A^{term} e^{\frac{r}{\beta}(T-t)} \equiv Q_A(T-t, q_A^{term})$$

Exhaustion of A requires

$$A = \int_0^T q_A(t)dt = \int_0^T q_A^{term} e^{\frac{r}{\beta}(T-t)} dt$$

Hence

$$A = \left(\frac{\beta}{r}\right) q_A^{term} \left[e^{(r/\beta)T} - 1\right]$$

Solving for T

$$T = \frac{\beta}{r} \ln\left[1 + \frac{rA}{\beta q_A^{term}}\right] = \frac{\beta}{r} \ln\left[1 + \frac{A}{p^{\frac{1}{1-\beta}} x}\right]$$

Notice that in this case, $T(.)$ is homogeneous of degree zero in (A, x).

Lemma 3 *The value function of Problem W is*

$$W(A; p, x) =$$

$$\int_0^{T(A, q_A^{term})} \left\{U(Q_A(T-t, q_A^{term})) - cQ_A(T-t, q_A^{term})\right\} e^{-rt} dt + e^{-rT(A, q_A^{term})} EV(X)$$

Example 1 (continued) Under CEMU and zero extraction costs, the integral of discounted utility flow obtained from layer A is

$$I(A) \equiv \int_0^T e^{-rt} \frac{1}{1-\beta} [q_A(t)]^{1-\beta} dt$$

$$= \int_0^T \frac{1}{1-\beta} \left\{\left[q_A^{term}\right] e^{\frac{r}{\beta}(T-t)}\right\}^{1-\beta} e^{-rt} dt$$

$$= \frac{\beta}{r(1-\beta)} \left[q_A^{term}\right]^{1-\beta} e^{-rT} \left[e^{(r/\beta)T} - 1\right] = \left[\frac{CS(q_A^{term})}{r}\right] e^{-rT} \left[e^{(r/\beta)T} - 1\right]$$

Thus, using (1)

$$\begin{aligned} W(A;p,x) &= I(A) + e^{-rT} EV(X) \\ &= I(A) + e^{-rT} \left[\frac{CS(q_A^{term})}{r}\right] \end{aligned}$$

This gives

$$\begin{aligned} W(A;p,x) &= \left[\frac{CS(q_A^{term})}{r}\right] e^{-rT} \left\{\left[e^{(r/\beta)T} - 1\right] + 1\right\} \\ &= \frac{CS(q_A^{term})}{r} \left[e^{\frac{rT}{\beta}}\right]^{1-\beta} \\ &= \frac{CS(q_A^{term})}{r} \left[1 + \frac{rA}{\beta q_A^{term}}\right]^{1-\beta} \end{aligned}$$

Simplifying,

$$W(A;p,x) = \left[\frac{q_A^{term} + (r/\beta)A}{q_A^{term}}\right]^{1-\beta} EV(X)$$

$$= \left[q_A^{term} + (r/\beta)A\right]^{1-\beta} \left[\frac{EV(X)}{(q_A^{term})^{1-\beta}}\right]$$

$$W(A;p,x) = \left[q_A^{term} + (r/\beta)A\right]^{1-\beta} \left(\frac{\beta}{r(1-\beta)}\right) \tag{6}$$

$$W(A;p,x) = \left(\frac{r}{\beta}\right)^{-\beta} \left[xp^{\frac{1}{1-\beta}} + A\right]^{1-\beta} \left(\frac{1}{1-\beta}\right) \tag{7}$$

It follows that a second layer of size x with probability p is "equivalent" to a second layer with size $y = xp^{\frac{1}{1-\beta}}$ obtained under certainty.

Remark (on the convexity of the value function) The value function (7) is linear in p if $A = 0$, but **strictly convex** in p if $A > 0$.

3. Two deposits, of which one is of unknown size

Now consider a scenario where there are two mines. Mine 1 has two layers, A_1 and X_1, where A_1 is a known number and X_1 is a random variable, with two possible values, x_1 and 0, with probabilities p_1 and $(1 - p_1)$. (Here x_1 is a known positive number.) The actual value taken by X_1 is known as soon as A_1 is exhausted. Mine 2 has only one layer, A_2. The marginal cost of extraction is c, which is identical for both mines.

It is easy to show that the optimal order of extraction is to exhaust A_1 first. After that, X_1 is known, and it is a matter of indifference whether to exhaust A_2 before extracting from X_1, or vice versa, or to have simultaneous extractions from A_2 and X_1. Intuitively, by extracting A_1 first (rather than A_2) one obtains information at an earlier date. This is valuable for decision making.

4. Two deposits, each of unknown size

Now suppose there are two mines, each of unknown size. Mine 1 is the same as described in the preceding section. Mine 2 has two layers, A_2 and X_2, where X_2 is a random variable that can take on one of two possible values x_2 or 0 (with probabilities p_2 and $1 - p_2$ respectively.) Except in singular cases, $p_1 \neq p_2$, $A_1 \neq A_2$ and $x_1 \neq x_2$. Assume $c_1 = c_2$. Under what condition would it be optimal to exhaust A_1 before extracting A_2?

To answer this question, it is useful to begin by determining the value of the program conditional on A_1 being extracted first.

5. Resolving the uncertainty about deposit 1 first

Suppose that the individual plans to exhaust A_1 first, and specifies some time t_1 at which the accumulated extraction from deposit 1 is A_1. Then, at time t_1, there are two possibilities: (i) Case G_1in which there is good news about deposit 1, i.e., $X_1 = x_1$, or (ii) Case B_1in which there is bad news about deposit 1, i.e., $X_1 = 0$.

5.1. Analysis of Case B_1

If case B_1 occurs then at t_1 the individual will begin to extract from deposit 2.We denote by W^{B_1} the value of this problem. Our earlier analysis of the one-deposit case (i.e. Problem W) applies. Let θ_2 denote the optimal length of time to exhaust A_2. Clearly θ_2 is a function of A_2 and $q_{A_2}^{termB_1}$, just as T (in Problem W) is a function of A and q_A^{term}. (Here $q_{A_2}^{termB_1}$ denotes the optimal terminal extraction rate for layer A_2, given that B_1 has occurred). After the exhaustion of A_2, there are either x_2 units of resources left, or none, because B_1 means $X_1 = 0$.

Clearly, from Proposition 1,

$$q_{A_2}^{termB_1} = CS^{-1}\left(rEV(X_2)\right)$$

and from Corollary 1,

$$\theta_2 = \theta_2(A_2, q_{A_2}^{termB_1}) = \theta_2(A_2, CS^{-1}\left(rEV(X_2)\right))$$

Hence

$$W^{B_1}(A_2, p_2, x_2) =$$

$$\int_0^{\theta_2(A_2, q_{A_2}^{termB_1})} \left\{U(Q_{A_2}(\theta_2 - \tau, q_{A_2}^{termB_1})) - cQ_{A_2}(\theta - \tau, q_{A_2}^{termB_1})\right\} e^{-r\tau} d\tau$$

$$+e^{-r\theta_2(A_2, q_{A_2}^{termB_1})} EV(X_2)$$

Lemma B1 *After receiving the bad news B_1 (i.e., that $X_1 = 0$) when A_1 is exhausted, the value of the remaining program is, under CEMU and zero extraction costs,*

$$W^{B_1} = \frac{CS(q_{A_2}^{termB_1})}{r}\left[1 + \frac{rA_2}{\beta q_{A_2}^{termB_1}}\right]^{1-\beta} =$$

$$\left[1+\frac{rA_2}{\beta q_{A_2}^{termB_1}}\right]^{1-\beta} EV(X_2+0)=\left(\frac{\beta}{r(1-\beta)}\right)\left[q_{A_2}^{termB_1}+\frac{r}{\beta}A_2\right]^{1-\beta}$$

where

$$q_{A_2}^{termB_1}=\left(\frac{r}{\beta}\right)p_2^{\frac{1}{1-\beta}}x_2$$

Thus

$$W^{B_1}=\frac{1}{1-\beta}\left(\frac{r}{\beta}\right)^{-\beta}\left[x_2 p_2^{\frac{1}{1-\beta}}+A_2\right]^{1-\beta}$$

The optimal time it takes to exhaust A_2 (given the bad news B_1) is

$$\theta_2^{B_1}=\frac{\beta}{r}\ln\left[1+\frac{rA_2}{\beta q_{A_2}^{termB_1}}\right]=\frac{\beta}{r}\ln\left[1+\frac{A_2}{p_2^{\frac{1}{1-\beta}}x_2}\right]$$

Proof Omitted.

5.2. *Analysis of Case G_1*

Case G_1 is slightly more complicated. The individual will also find it optimal to begin to extract from deposit 2, knowing that, at the time the layer A_2 is exhausted, he will receive news whether he has x_1+x_2 (we call this sub-case G_1G_2) or has only x_1 left (we call this sub-case G_1B_2). This is different from Case B_1 where after exhausting A_2 he will have either x_2 units left, or none. Clearly, $q_{A_2}^{termG_1}$ is different from $q_{A_2}^{termB_1}$. A little reflection reveals that

$$q_{A_2}^{termG_1}=CS^{-1}\left(rEV(X_2+x_1)\right)$$

where X_2 is a random variable, and x_1 is a known constant (not a random variable) because G_1 has occured.

Given that G_1 has occurred, the value of the sub-case G_1G_2 (once A_2 has been exhausted) is, in the CEMU case,

$$J^{G_1G_2}\equiv V(x_1+x_2)=\frac{(x_1+x_2)^{1-\beta}}{1-\beta}\left(\frac{r}{\beta}\right)^{-\beta}$$

and the value of the sub-case G_1B_2 (once A_2 has been exhausted) is

$$J^{G_1B_2} \equiv V(x_1) = \frac{(x_1)^{1-\beta}}{1-\beta}\left(\frac{r}{\beta}\right)^{-\beta}$$

Thus

$$EV(X_2 + x_1) = p_2 J^{G_1G_2} + (1-p_2)J^{G_1B_2}$$

$$EV(X_2 + x_1) = \left(\frac{1}{1-\beta}\right)\left(\frac{r}{\beta}\right)^{-\beta}\left[p_2(x_1+x_2)^{1-\beta} + (1-p_2)x_1^{1-\beta}\right]$$

The individual would be indifferent between this expected utility and receiving a certain stock z_2 where

$$z_2^{1-\beta} \equiv p_2(x_1+x_2)^{1-\beta} + (1-p_2)x_1^{1-\beta}$$

i.e., z_2 is the **certainty-equivalent** of the prospect $(x_1+x_2, x_1; p_2, 1-p_2)$.In case G_1, the individual must decide on the length $\theta_2^{G_1}$ of the time interval over which he must use up the layer A_2.Given t_1 and G_1, his optimization problem is

Problem G_1 (After receiving the **good news** that $X_1 = x_1 > 0$): Find $\theta_2^{G_1}$ and the time path $q_{A_2}^{G_1}(.)$ to maximize W^{G_1} defined by

$$\int_{t_1}^{t_1+\theta_2^{G_1}} e^{-r(t-t_1)}\left[U(q_{A_2}^{G_1}(t)) - cq_{A_2}^{G_1}(t)\right]dt$$

$$+e^{-r\theta_2^{G_1}}\left[p_2V(x_1+x_2) + (1-p_2)V(x_1)\right]$$

subject to

$$\int_{t_1}^{t_1+\theta_2^{G_1}} q_{A_2}(t)dt = A_2$$

Applying Proposition 1 and Corollary1, we get the following results.

Proposition 2 *The solution to Problem G_1 consists of (i) a terminal extraction rate for layer A_2, denoted by $q_{A_2}^{termG_1}$, (ii) an optimal time $\theta_2^{G_1}$ and (iii) a time path $q_{A_2}^{G_1}(t)$ over the time interval $\left[t_1, t_1+\theta_2^{G_1}\right]$, with the properties that*

1) $q_{A_2}^{termG_1}$ *satisfies the condition that the consumer's surplus at* $q_{A_2}^{termG_1}$ *is equated to the product of the interest rate* r *and the social value of the random variable* $X_2 + x_1$ *(here* x_1 *is a known number), that is,*

$$U(q_{A_2}^{termG_1}) - U'(q_{A_2}^{termG_1})q_{A_2}^{termG_1} = r\left[p_2 V(x_1 + x_2) + (1 - p_2)V(x_1)\right] \tag{8}$$

2) over the time interval $\left[t_1, t_1 + \theta_2^{G_1}\right]$, *the extraction path satisfies the Hotelling Rule*

$$U'(q_{A_2}(t))e^{-rt} = U'(q_{A_2}^{termG_1})e^{-r(t_1+\theta_2)}$$

3) the path $q_{A_2}(t)$ *over* $\left[t_1, t_1 + \theta_2^{G_1}\right]$ *just exhausts the first layer,* A_2, that is,

$$\int_{t_1}^{t_1+\theta_2^{G_1}} q_{A_2}^{G_1}(t)dt = A_2$$

Proof Similar to that of Proposition 1, and is therefore omitted.

Applying Proposition 2 to the CEMU case, we get

$$\frac{\beta}{1-\beta}\left[q_{A_2}^{termG_1}\right]^{1-\beta} = r\left[p_2\frac{(x_1+x_2)^{1-\beta}}{1-\beta}\left(\frac{r}{\beta}\right)^{-\beta} + (1-p_2)\frac{(x_1)^{1-\beta}}{1-\beta}\left(\frac{r}{\beta}\right)^{-\beta}\right]$$

$$\left[q_{A_2}^{termG_1}\right]^{1-\beta} = \left(\frac{r}{\beta}\right)^{1-\beta}\left[p_2(x_1+x_2)^{1-\beta} + (1-p_2)(x_1)^{1-\beta}\right]$$

$$q_{A_2}^{termG_1} = \left(\frac{r}{\beta}\right)\left[p_2(x_1+x_2)^{1-\beta} + (1-p_2)(x_1)^{1-\beta}\right]^{1/(1-\beta)} \equiv \frac{rz_2}{\beta} \tag{9}$$

Notice that $q_{A_2}^{termG_1}$ is homogeneous of degree one in (x_1, x_2).

Corollary G1 *After receiving the good news* G_1*(i.e., that* $X_1 = x_1$*) when* A_1 *is exhausted, the value of the remaining program, under CEMU and zero extraction costs, is*

$$W^{G_1} = \left(\frac{\beta}{r(1-\beta)}\right)\left[q_{A_2}^{termG_1} + \frac{r}{\beta}A_2\right]^{1-\beta}$$

where

$$q_{A_2}^{termG_1} = \left(\frac{r}{\beta}\right)\left[p_2(x_1+x_2)^{1-\beta} + (1-p_2)(x_1)^{1-\beta}\right]^{1/(1-\beta)} \equiv \left(\frac{r}{\beta}\right) z_2$$

i.e.

$$W^{G_1} = \frac{1}{1-\beta}\left(\frac{r}{\beta}\right)^{-\beta}[z_2 + A_2]^{1-\beta}$$

Proof: Omitted.

Remark: Notice that W^{G_1} is homogeneous of degree $1-\beta$ in (A_2, x_1, x_2).

5.3. *Optimal time to exhaust A_1*

We now compute the optimal time t_1, given that at time zero the individual chooses to extract from layer A_1 first. The optimization problem is

Problem $A1$ Choose the time t_1 and the extraction path $\widehat{q}_{A_1}(t)$ over $[0, t_1]$ to maximize

$$\int_0^{t_1} e^{-rt}\left[U(\widehat{q}_{A_1}(t)) - c\widehat{q}_{A_1}(t)\right]dt + e^{-rt_1}\left[p_1 W^{G_1} + (1-p_1)W^{B_1}\right]$$

subject to

$$\int_0^{t_1} \widehat{q}_{A_1}(t)dt = A_1$$

Remark We use the symbol $\widehat{q}_{A_1}(t)$ (with the hat) to denote that the path is chosen at time 0, when the individual *has not received any news* (good or bad) about any of the deposits.

Proposition 3 *The solution of Problem A_1 consists of (i) a terminal extraction rate for layer A_1, denoted by $\widehat{q}_{A_1}^{term}$, (ii) an optimal time t_1 and (iii) a time path $\widehat{q}_{A_1}(t)$ over the time interval $[0, t_1]$, with the following properties.*

1) $\widehat{q}_{A_1}^{term}$ satisfies the condition that the consumer's surplus at $\widehat{q}_{A_1}^{term}$ is equated to the product of the interest rate r and the social value of the *random variables X_2 and X_1 (here X_1 is unknown, because in choosing the path $\widehat{q}_{A_1}(t)$ at time 0, the layer A_1 has not been exhausted).*

$$U(\widehat{q}_{A_1}^{term}) - U'(\widehat{q}_{A_1}^{term})\widehat{q}_{A_1}^{term} = r\left[p_1 W^{G_1} + (1-p_1)W^{B_1}\right]$$

2) Over the time interval $[0, t_1]$, the extraction path satisfies the Hotelling Rule:

$$U'(\widehat{q}_1(t))e^{-rt} = U'(\widehat{q}_{A_1}^{term})e^{-rt_1}$$

3) The path $\widehat{q}_{A_1}(t)$ over $[0, t_1]$ just exhausts the first layer A_1 so that

$$\int_0^{t_1} \widehat{q}_{A_1}(t)dt = A_1.$$

Corollary 3 *Given CEMU and zero costs of extraction, the values $\widehat{q}_{A_1}^{term}$ and t_1 can be computed as follows.*

$$\frac{\beta}{1-\beta}\left[\widehat{q}_{A_1}^{term}\right]^{1-\beta} = r\left[p_1 W^{G_1} + (1-p_1)W^{B_1}\right]$$

where

$$W^{G_1} \equiv \frac{1}{1-\beta}\left(\frac{\beta}{r}\right)^{\beta}\left\{\left[p_2(x_1+x_2)^{1-\beta} + (1-p_2)x_1^{1-\beta}\right]^{\frac{1}{1-\beta}} + A_2\right\}^{1-\beta}$$

$$W^{B_1} \equiv \frac{1}{1-\beta}\left(\frac{\beta}{r}\right)^{\beta}\left\{\left[p_2(x_2)^{1-\beta}\right]^{\frac{1}{1-\beta}} + A_2\right\}^{1-\beta}$$

Thus, if we define the "quantity indices"

$$Y^{G_1} \equiv \left[p_2(x_1+x_2)^{1-\beta} + (1-p_2)x_1^{1-\beta}\right]^{\frac{1}{1-\beta}} + A_2$$

$$Y^{B_1} \equiv \left[p_2(x_2)^{1-\beta}\right]^{\frac{1}{1-\beta}} + A_2$$

then

$$\widehat{q}_{A_1}^{term} = \left(\frac{r}{\beta}\right)\left[p_1\left(Y^{G_1}\right)^{1-\beta} + (1-p_1)\left(Y^{B_1}\right)^{1-\beta}\right]^{\frac{1}{1-\beta}}$$

Remark: Notice that $\widehat{q}_{A_1}^{term}$ is homogeneous of degree one in (A_2, x_1, x_2).

Under CEMU and zero extraction costs, the optimal t_1 is determined from the condition

$$A_1 = \int_0^{t_1} \widehat{q}_1(t)dt = \widehat{q}_{A_1}^{term} \left(\frac{\beta}{r}\right) \left[e^{(r/\beta)t_1} - 1\right]$$

Hence

$$t_1 = \frac{\beta}{r} \ln \left[1 + \frac{(r/\beta)A_1}{\widehat{q}_{A_1}^{term}}\right]$$

The value of the integral of utility over the time interval $[0, t_1]$ is

$$I(A_1; A_2, x_1, x_2) = \frac{1}{(1-\beta)} \left[\widehat{q}_{A_1}^{term}\right]^{1-\beta} e^{-rt_1} \left[e^{(r/\beta)t_1} - 1\right] \left(\frac{\beta}{r}\right)$$

and the value of the program, given that A_1 is to be exhausted first, is

$$W^{A_1} = I(A_1; A_2, x_1, x_2) + e^{-rt_1} \left[p_1 W^{G_1} + (1-p_1) W^{B_1}\right]$$

$$= \left(\frac{\beta}{r(1-\beta)}\right) \left[\widehat{q}_{A_1}^{term} + \frac{r}{\beta} A_1\right]^{1-\beta}$$

$$= \frac{1}{1-\beta} \left(\frac{r}{\beta}\right)^{-\beta} \left\{\left[p_1 \left(Y^{G_1}\right)^{1-\beta} + (1-p_1) \left(Y^{B_1}\right)^{1-\beta}\right]^{\frac{1}{1-\beta}} + A_1\right\}^{1-\beta}$$

6. Which deposit to extract first?

Clearly, by similar reasoning, if we extract from layer A_2 first, the welfare level will be

$$W^{A_2} = \left(\frac{\beta}{r(1-\beta)}\right) \left[\widehat{q}_{A_2}^{term} + \frac{r}{\beta} A_2\right]^{1-\beta}$$

where

$$\widehat{q}_{A_2}^{term} = \left(\frac{r}{\beta}\right) \left[p_2 \left(Y^{G_2}\right)^{1-\beta} + (1-p_2) \left(Y^{B_2}\right)^{1-\beta}\right]^{\frac{1}{1-\beta}}$$

$$Y^{G_2} \equiv \left[p_1(x_1+x_2)^{1-\beta} + (1-p_1)x_2^{1-\beta}\right]^{\frac{1}{1-\beta}} + A_1$$

$$Y^{B_2} \equiv \left[p_1(x_1)^{1-\beta}\right]^{\frac{1}{1-\beta}} + A_1$$

To determine whether to extract A_1 or A_2 first, we must compare W^{A_2} with W^{A_1}.

Clearly, A_1 must be extracted before A_2 if and only if the following ratio is greater than unity

$$\rho \equiv \frac{\widehat{q}_{A_1}^{term} + \frac{r}{\beta}A_1}{\widehat{q}_{A_2}^{term} + \frac{r}{\beta}A_2}$$

i.e.,

$$\widehat{q}_{A_2}^{term} + \frac{r}{\beta}A_2 < \widehat{q}_{A_1}^{term} + \frac{r}{\beta}A_1$$

i.e.,

$$\Delta \equiv \left[p_2\left(Y^{G_2}\right)^{1-\beta} + (1-p_2)\left(Y^{B_2}\right)^{1-\beta}\right]^{\frac{1}{1-\beta}} + A_2$$
$$-\left[p_1\left(Y^{G_1}\right)^{1-\beta} + (1-p_1)\left(Y^{B_1}\right)^{1-\beta}\right]^{\frac{1}{1-\beta}} - A_1 < 0 \qquad (10)$$

We consider three special cases.

Case 1 $A_1 = A_2 = A$, $x_1 = x_2 = x$, $p_1 > p_2$

In this case, it is optimal to extract A_1 first. (See the Appendix for a proof.) The intuition is that one would want to have news about the better propect first.

Case 2 $p_1 = p_2 = p$, $x_1 = x_2$, $A_1 > A_2$.

In this case, it is optimal to extract A_2 first. (See the Appendix for a proof.) The intuition is that A_2 gives news at an earlier date that A_1.

Case 3 $A_1 = A_2 = A$, $p_1 = p_2 = p$, $x_1 > x_2$. In this case, it is optimal to extract A_1 first.

We summarize our results in the following proposition:

Proposition 4 (**Optimal order of extraction of two deposits of unknown sizes**) *In the case of two deposits of unknown sizes, the optimal policy has the following properties:*

a) Other things being equal, the first layer of the deposit with higher probability of x should be extracted (and exhausted) first.

b) Other things being equal, the first layer of the deposit with a smaller first layer should be extracted (and exhausted) first.

c) Other things being equal, the first layer of the deposit with a higher x should be extracted (and exhausted) first.

Proof: See the Appendix.

7. Extension: Learning while extracting

So far we have assumed that the individual learns about the second layer (i.e., finds out if it exists or not) only after the exhaustion of the first layer. It would be a bit more realistic to suppose, instead, that some information arrives while the individual is in the process of extracting the first layer. A simple way of modelling this is as follows. The individual's subjective probability numbers, p_1 and p_2, are only ex-ante, or preliminary, probabilities. When the individual is in the process of extracting layer A_i, he receives news that allow him to update his p_i. To make things as simple as possible, suppose that, for deposit i, there exists a number α_i (where $0 < \alpha_i < 1$) such that after the fraction α_i of A_i is used up, he will be able to revise p_i upwards, to $p_i + \delta_i$, or downwards, to $p_i - \delta_i$. (We restrict δ_i so that $0 < p_i - \delta_i \leq p_i + \delta_i < 1$.) Before $\alpha_i A_i$is exhausted, he does not know if the revision is going to be upwards, or downwards. He only knows that the probability of upward revision is π_i and that of downward revision is $1 - \pi_i$.

How should the individual proceed? How fast should he extract the first fraction $\alpha_i A_i$? Let us begin with the case of a single deposit with two layers.

7.1. *One deposit with learning while extracting*

The deposit has two layers. The size of the first layer is A, which is known. The size of the second layer is either x or 0. Ex ante, the probability that $X = x$ is p and the probability that $X = 0$ is $1 - p$. We represent this "prospect" by the tuple $(x, 0; p, 1-p)$. As soon as a fraction $\alpha \in (0, 1)$ of A is exhausted, the decision maker obtains news about the second layer. (We assume the number α is known to the individual.) The news is "bad" if p must be downgraded to $p-\delta$. This case is denoted by b (for "bad"). It is "good" if p must be upgraded to $p + \delta$. This case is denoted by g (for "good").

We must determine the decision maker's optimal decision at node g and at node b. In what follows, we focus on the case of CEMU utility with zero extraction costs. It is clear that our earlier analysis applies here, with minor modifications.

At node g, the remaining part of the first layer is $(1 - \alpha)A$. After he has learned the good news, the decision maker's problem is to maximize

$$\int_0^{T^g} U(q)e^{-rt}dt + e^{-rT^g}\left[(p + \delta)V(x)\right]$$

subject to

$$\int_0^{T^g} q(t)dt = (1 - \alpha)A \equiv \gamma A$$

Here T^g denotes the optimal length of time to consume the "second half" of the first layer, i.e., γA.(For convenience, we use the expression "second half" to denote γA, which is in general not $A/2$).

The solution can be characterized by (a) the terminal extraction rate of the "second half "of the first layer, given the good news g,

$$q_{\gamma A}^{termg} = \left(\frac{r}{\beta}\right) x(p + \delta)^{\frac{1}{1-\beta}}$$

and by (b) T^g, the length of time to extract the "second half"

$$T^g = \frac{\beta}{r} \ln\left[1 + \frac{r\gamma A}{\beta q_{\gamma A}^{termg}}\right] = \frac{\beta}{r} \ln\left[1 + \frac{\gamma A}{(p + \delta)^{\frac{1}{1-\beta}} x}\right]$$

The welfare (from time T^g) as seen at node g is

$$V^g = W(\gamma A; p + \delta, x) = \left(\frac{r}{\beta}\right)^{-\beta} \left[x(p + \delta)^{\frac{1}{1-\beta}} + \gamma A\right]^{1-\beta} \left(\frac{1}{1 - \beta}\right)$$

Similarly, at node b,

$$T^b = \frac{\beta}{r} \ln \left[1 + \frac{r\gamma A}{\beta q_{\gamma A}^{termb}} \right] = \frac{\beta}{r} \ln \left[1 + \frac{\gamma A}{(p-\delta)^{\frac{1}{1-\beta}} x} \right]$$

where the terminal extraction rate of the "second half "of the first laye first layer, given bad news, is

$$q_{\gamma A}^{termb} = \left(\frac{r}{\beta} \right) x (p-\delta)^{\frac{1}{1-\beta}}$$

The welfare as seen at node b is

$$V^b = W(\gamma B; p-\delta, x) = \left(\frac{r}{\beta} \right)^{-\beta} \left[x(p-\delta)^{\frac{1}{1-\beta}} + \gamma B \right]^{1-\beta} \left(\frac{1}{1-\beta} \right)$$

We must now compute the optimal extraction of the "first half " of A. The problem to be solved is:

$$\max \int_0^{T^*} U(c) e^{-rt} dt + e^{-rT^*} (\pi V^g + (1-\pi) V^b)$$

subject to

$$\int_0^{T^*} q(t) dt = \alpha A$$

where T^* is the optimal time to exhaust the "first half" of layer A.

The optimal terminal extraction rate for the "first half" of the layer A is obtained from

$$\frac{\beta}{1-\beta} \left[q_{\alpha A}^{term} \right]^{1-\beta} = r(\pi V^g + (1-\pi) V^b)$$

Define the quantity index

$$\Omega(\pi, p, \delta, x, \gamma A) \equiv \left\{ \pi \left[x(p+\delta)^{\frac{1}{1-\beta}} + \gamma A \right]^{1-\beta} + (1-\pi) \left[x(p-\delta)^{\frac{1}{1-\beta}} + \gamma A \right]^{1-\beta} \right\}^{\frac{1}{1-\beta}} \tag{11}$$

where, in view of

$$\Omega(\pi, p, \delta, x, o) = x(p)^{\frac{1}{1-\beta}}$$

Then the optimal terminal extraction rate for the "first half "of layer A is

$$q_{\alpha A}^{term} = \frac{r}{\beta}\Omega$$

and the time it takes to exhaust the first half of layer A is

$$T^* = \frac{\beta}{r}\ln\left[1 + \frac{r\alpha A}{\beta q_{\alpha A}^{term}}\right] = \frac{\beta}{r}\ln\left[1 + \frac{\alpha A}{\Omega}\right]$$

The optimal value of the whole program is

$$J(A; \pi, p, \delta, x, \alpha) = \left[\frac{q_{\alpha A}^{term} + (r/\beta)\alpha A}{q_{\alpha A}^{term}}\right]^{1-\beta}(\pi V^g + (1-\pi)V^b)$$

$$= \left[\frac{q_{\alpha A}^{term} + (r/\beta)\alpha A}{q_{\alpha A}^{term}}\right]^{1-\beta}\Omega^{1-\beta}\left(\frac{r}{\beta}\right)^{-\beta}\left(\frac{1}{1-\beta}\right)$$

$$= [\Omega + \alpha A]^{1-\beta}\left(\frac{r}{\beta}\right)^{-\beta}\left(\frac{1}{1-\beta}\right)$$

Note that

$$\Omega(\pi, p, 0, x, \gamma A) = xp^{\frac{1}{1-\beta}} + \gamma A$$

Let us determine whether the optimal value $J(A; \pi, p, \delta, x, \alpha)$ is higher or lower than the value obtained if there is no expectation of mid-way revision of p. Clearly, if $\delta = 0$ then the values of the two problems are identical:

$$J(A; \pi, p, 0, x, \alpha) = W(A; p, x) = \left(\frac{r}{\beta}\right)^{-\beta}\left[xp^{\frac{1}{1-\beta}} + A\right]^{1-\beta}\left(\frac{1}{1-\beta}\right)$$

The sign of the derivative of $J(A; \pi, p, \delta, x, \alpha)$ with respect to δ (keeping π constant) is the same as the sign of the following expression:

$$\phi(\pi, \delta, \gamma) \equiv \pi\left[x(p+\delta)^{\frac{1}{1-\beta}} + \gamma A\right]^{-\beta}(p+\delta)^{\frac{\beta}{1-\beta}} - (1-\pi)\left[x(p-\delta)^{\frac{1}{1-\beta}} + \gamma A\right]^{-\beta}(p-\delta)^{\frac{\beta}{1-\beta}}$$

where $\phi(\pi, 0, \gamma) = 0$. Note that for $\delta > 0$, $\phi(\pi, \delta, \gamma) > 0$ if and only if

$$\frac{\pi}{1-\pi}\left[\left(\frac{x(p-\delta)^{\frac{1}{1-\beta}}+\gamma A}{x(p+\delta)^{\frac{1}{1-\beta}}+\gamma A}\right)\left(\frac{p+\delta}{p-\delta}\right)^{\frac{1}{1-\beta}}\right]^{\beta} > 1$$

i.e., iff

$$\frac{x\left[(p-\delta)(p+\delta)\right]^{\frac{1}{1-\beta}}+\gamma A(p+\delta)^{\frac{1}{1-\beta}}}{x\left[(p-\delta)(p+\delta)\right]^{\frac{1}{1-\beta}}+\gamma A(p-\delta)^{\frac{1}{1-\beta}}} > \left(\frac{1-\pi}{\pi}\right)^{\frac{1}{\beta}} \tag{12}$$

If $\pi = 1 - \pi$, the inequality (12) is satisfied for all positive δ and γ such that $0 < p - \delta < p + \delta < 1$, and $0 < \gamma < 1$.

Proposition 5 (gain from learning while extracting) *The possibility of learning while extracting increases the welfare of the individual (compared with the no-learning scenario).*

7.2. Two deposits with learning while extracting

Consider now the case of two deposits with learning while extracting. For simplicity, assume $\alpha = 1/2$. Deposit i consists of a layer A_i and a second layer of size X_i where X_i is a random variable that can take value x_i (a known number) with ex-ante probability p_i, or zero with ex-ante probability $1 - p_i$. Extracting the first half of layer A_i gives information that allows revision of p_i upwards to $p_i + \delta_i$ or downwards to $p_i - \delta_i$. We assume that the decision maker learns nothing about one deposit by extracting another deposit. In that sense the deposits are assumed to be very dissimilar. What is the optimal order of extraction? We suppose that it is feasible to extract the first half of A_i, and then costlessly switch to the extraction of the first half of A_j where $j \neq i$. Such a strategy is called “midway switching”. Is it ever optimal to do midway switching?

We maintain the assumption that it is not possible to extract from the second layer of a deposit before exhausting its first layer. (That is, the second layer is not accessible before the first layer is removed).

And we take it as self-evident that it is never optimal to begin extracting from an accessible second layer before all first layers have been exhausted.

How should the first layers be extracted? In what follows, we assume $\alpha_i = 1/2$ for all i. We list below six possible patterns of extraction

There are two extraction patterns with midway switching in an inter-weaving mode:

$$\text{Pattern 1}\left(\frac{A_1}{2},\frac{A_2}{2},\frac{A_1}{2},\frac{A_2}{2}\right)$$

$$\text{Pattern 2}\left(\frac{A_2}{2},\frac{A_1}{2},\frac{A_2}{2},\frac{A_1}{2}\right)$$

There are two extraction patterns with midway switching in a bunching mode:

$$\text{Pattern 3}\left(\frac{A_1}{2},\frac{A_2}{2},\frac{A_2}{2},\frac{A_1}{2}\right)$$

$$\text{Pattern 4}\left(\frac{A_2}{2},\frac{A_1}{2},\frac{A_1}{2},\frac{A_2}{2}\right)$$

And there are two extraction patterns without midway switching:

$$\text{Pattern 5}\left(\frac{A_1}{2},\frac{A_1}{2},\frac{A_2}{2},\frac{A_2}{2}\right)$$

$$\text{Pattern 6}\left(\frac{A_2}{2},\frac{A_2}{2},\frac{A_1}{2},\frac{A_1}{2}\right)$$

These patterns, however, are not strategies. By definition, strategies are conditional on information received at each node.

Let us simplify by assuming, for the moment, that the first half of A_1 must be exhausted first. Then the first information received is whether p_1 should be upgraded (the news is g_1) or downgraded (the news is b_1). Whether the news is good (g_1) or bad (b_1) the next choice is whether to extract the second half of A_1 or the first half of A_2. Denote these choices by A_1^s (meaning the second half of A_1) and A_2^f (meaning the first half of A_2). One can construct a game tree, with many branches. The number of strategies is very large, even on the assumption that we begin with the first half of A_1 so that the first piece of information is either g_1 or b_1. Below are a few possible strategies, given A_1^f (meaning that the first half of A_1 must be exhausted first).

Strategy 1

$$g_1 \to A_2^f(g_1g_2 \to A_1^s; g_1b_2 \to A_1^s); b_1 \to A_2^f(b_1g_2 \to A_1^s; b_1b_2 \to A_1^s)$$

This strategy says that if the outcome of A_1^f is g_1, then the next step is to extract the first half of layer A_2, while if the outcome of A_1^f is b_1, then the next step is also to extract the first half of layer A_2; the first two observations can be g_1g_2, or g_1b_2, or b_1g_2, or b_1b_2. If g_1g_2 is observed, then (after extracting A_1^f and A_2^f) extract the second half of A_1 (i.e., choose A_1^s), etc.

Strategy 2

$$g_1 \to A_2^f(g_1g_2 \to A_1^s; g_1b_2 \to A_1^s); b_1 \to A_2^f(b_1g_2 \to A_1^s; b_1b_2 \to A_2^s)$$

This strategy differs from strategy 1 only in the last entry: after observing b_1b_2, choose A_2^s (and not A_1^s).

Strategy 3

$$g_1 \to A_2^f(g_1g_2 \to A_1^s; g_1b_2 \to A_1^s); b_1 \to A_2^f(b_1g_2 \to A_2^s; b_1b_2 \to A_1^s)$$

Strategy 4

$$g_1 \to A_2^f(g_1g_2 \to A_1^s; g_1b_2 \to A_1^s); b_1 \to A_2^f(b_1g_2 \to A_2^s; b_1b_2 \to A_2^s)$$

Strategy 5

$$g_1 \rightarrow A_2^f(g_1g_2 \rightarrow A_1^s; g_1b_2 \rightarrow A_2^s); b_1 \rightarrow A_2^f(b_1g_2 \rightarrow A_1^s; b_1b_2 \rightarrow A_1^s)$$

Strategy 6

$$g_1 \rightarrow A_2^f(g_1g_2 \rightarrow A_1^s; g_1b_2 \rightarrow A_2^s); b_1 \rightarrow A_2^f(b_1g_2 \rightarrow A_1^s; b_1b_2 \rightarrow A_2^s)$$

Strategy 7

$$g_1 \rightarrow A_2^f(g_1g_2 \rightarrow A_1^s; g_1b_2 \rightarrow A_2^s); b_1 \rightarrow A_2^f(b_1g_2 \rightarrow A_2^s; b_1b_2 \rightarrow A_1^s)$$

Strategy 8

$$g_1 \rightarrow A_2^f(g_1g_2 \rightarrow A_1^s; g_1b_2 \rightarrow A_2^s); b_1 \rightarrow A_2^f(b_1g_2 \rightarrow A_2^s; b_1b_2 \rightarrow A_2^s)$$

Strategy 9

$$g_1 \rightarrow A_2^f(g_1g_2 \rightarrow A_2^s; g_1b_2 \rightarrow A_1^s); b_1 \rightarrow A_2^f(b_1g_2 \rightarrow A_1^s; b_1b_2 \rightarrow A_1^s)$$

Strategy 10

$$g_1 \rightarrow A_2^f(g_1g_2 \rightarrow A_2^s; g_1b_2 \rightarrow A_1^s); b_1 \rightarrow A_2^f(b_1g_2 \rightarrow A_1^s; b_1b_2 \rightarrow A_2^s)$$

Strategy 11

$$g_1 \rightarrow A_2^f(g_1g_2 \rightarrow A_2^s; g_1b_2 \rightarrow A_1^s); b_1 \rightarrow A_2^f(b_1g_2 \rightarrow A_2^s; b_1b_2 \rightarrow A_1^s)$$

Strategy 12

$$g_1 \rightarrow A_2^f(g_1g_2 \rightarrow A_2^s; g_1b_2 \rightarrow A_1^s); b_1 \rightarrow A_2^f(b_1g_2 \rightarrow A_2^s; b_1b_2 \rightarrow A_2^s)$$

Strategy 13

$$g_1 \to A_2^f(g_1g_2 \to A_2^s; g_1b_2 \to A_2^s); b_1 \to A_2^f(b_1g_2 \to A_1^s; b_1b_2 \to A_1^s)$$

Strategy 14

$$g_1 \to A_2^f(g_1g_2 \to A_2^s; g_1b_2 \to A_2^s); b_1 \to A_2^f(b_1g_2 \to A_1^s; b_1b_2 \to A_2^s)$$

Strategy 15

$$g_1 \to A_2^f(g_1g_2 \to A_2^s; g_1b_2 \to A_2^s); b_1 \to A_2^f(b_1g_2 \to A_2^s; b_1b_2 \to A_1^s)$$

Strategy 16

$$g_1 \to A_2^f(g_1g_2 \to A_2^s; g_1b_2 \to A_2^s); b_1 \to A_2^f(b_1g_2 \to A_2^s; b_1b_2 \to A_2^s)$$

Strategy 17

$$g_1 \to A_1^s(g_1G_1 \to A_2^f; g_1B_1 \to A_2^f); b_1 \to A_2^f(b_1g_2 \to A_1^s; b_1b_2 \to A_1^s)$$

Strategy 18

$$g_1 \to A_1^s(g_1G_1 \to A_2^f; g_1B_1 \to A_2^f); b_1 \to A_2^f(b_1g_2 \to A_1^s; b_1b_2 \to A_2^s)$$

Strategy 19

$$g_1 \to A_1^s(g_1G_1 \to A_2^f; g_1B_1 \to A_2^f); b_1 \to A_2^f(b_1g_2 \to A_2^s; b_1b_2 \to A_1^s)$$

Strategy 20

$$g_1 \to A_1^s(g_1G_1 \to A_2^f; g_1B_1 \to A_2^f); b_1 \to A_2^f(b_1g_2 \to A_2^s; b_1b_2 \to A_2^s)$$

Strategy 21

$$g_1 \to A_1^s(g_1G_1 \to A_2^f; g_1B_1 \to A_2^f); b_1 \to A_1^s(b_1G_1 \to A_2^f; b_1G_1 \to A_2^s)$$

While in principle it is possible to compute the expected payoff of each of these strategies, the analytical expressions become very cumbersome.

8. Concluding remarks

Our analysis of optimal order of exploitation under uncertainty can be generalised in several directions. First, we can introduce correlations across deposits. Second, extraction costs may differ across deposits. Then it is possible that the optimal extraction plan requires a high cost layer A_1 of deposit 1 to be exhausted before extracting a lower cost layer A_2 of deposit 2, because the value of information obtained from extracting A_1 may be higher than A_2.

Acknowledgements: We thank SSHRC and FQRSC for research grants, and Gérard Gaudet, John Hartwick, Bjarne S. Jensen, Takashi Kunimoto, Pierre Lasserre, Daniel Léonard, and Kim Long for comments and discussions.

APPENDIX

Proof of Proposition 4

Case 1 $A_1 = A_2 = A$, $x_1 = x_2 = x$, $p_1 \geq p_2$.

Let $p_1 = p + \varepsilon$, with $\varepsilon \geq 0$ and $p_2 = p$. If $\varepsilon = 0$, it is a matter of indifference whether A_2 or A_1 is exhausted first. Let us start at $\varepsilon = 0$, then increase ε to a small positive number. This means that, as ε increases, the second layer of mine 1 becomes more likely to exist than the second layer of mine 2. In this case, do we want to have news about mine 1 before news about mine 2? This would be the case if $\Delta(\varepsilon)$, defined below, is negative for small $\varepsilon > 0$. (Conversely, if $\Delta(\varepsilon) > 0$ for small $\varepsilon > 0$ then we should extract A_2 first.)

$$\Delta(\varepsilon) \equiv p\left(x\left[(p+\varepsilon)(2)^{1-\beta} + (1-p_1-\varepsilon)\right]^{\frac{1}{1-\beta}} + A\right)^{1-\beta} + (1-p)\left(x(p+\varepsilon)^{\frac{1}{1-\beta}} + A\right)^{1-\beta}$$

$$-(p+\varepsilon)\left(x\left[p(2)^{1-\beta} + (1-p)\right]^{\frac{1}{1-\beta}} + A\right)^{1-\beta} - (1-p-\varepsilon)\left(xp^{\frac{1}{1-\beta}} + A\right)^{1-\beta}$$

Clearly $\Delta(\varepsilon) < 0$ for small $\varepsilon > 0$ if $\Delta'(\varepsilon) < 0$ at $\varepsilon = 0$.

Let us calculate this derivative

$$\Delta'(\varepsilon) = \frac{px\left[(p+\varepsilon)(2)^{1-\beta} + (1-p_1-\varepsilon)\right]^{\beta/(1-\beta)}\left[(2)^{1-\beta} - 1\right]}{\left(x\left[(p+\varepsilon)(2)^{1-\beta} + (1-p_1-\varepsilon)\right]^{\frac{1}{1-\beta}} + A\right)^{\beta}}$$

$$+\frac{(1-p)x\left[p+\varepsilon\right)\right]^{\frac{\beta}{1-\beta}}}{\left(x(p+\varepsilon)^{\frac{1}{1-\beta}} + A\right)^{\beta}} -$$

$$-\left\{\left(x\left[p(2)^{1-\beta} + (1-p)\right]^{\frac{1}{1-\beta}} + A\right)^{1-\beta} + \left(x\left[p\right]^{\frac{1}{1-\beta}} + A\right)^{1-\beta}\right\}$$

Note that the term inside the curly brackets is positive. We can show that $\Delta'(\varepsilon) < 0$ at $\varepsilon = 0$ if A is small enough. To see this, let $A \to 0$, then $\Delta'(0)$ tends to

$$px^{1-\beta}\left[(2)^{1-\beta} - 1\right] + (1-p)x^{1-\beta} - x^{1-\beta}\left\{p(2)^{1-\beta} + 1\right\} \tag{13}$$

which has the sign of

$$-p - 1 + (1 - p) = -2p < 0$$

Case 2 $p_1 = p_2 = p$, $x_1 = x_2$, $A_1 > A_2$.
We will show that $\Delta \geq 0$, i.e.

$$\left[p\left(u+A_1\right)^{1-\beta}+(1-p)\left(v+A_1\right)^{1-\beta}\right]^{1/(1-\beta)}-\left[p\left(u+A_2\right)^{1-\beta}+(1-p)\left(v+A_2\right)^{1-\beta}\right]^{1/(1-\beta)} \geq A_1 - A_2 \qquad (14)$$

where

$$u \equiv x\left[p(2)^{1-\beta}+(1-p)\right]^{\frac{1}{1-\beta}}$$

$$v \equiv x\left[p\right]^{\frac{1}{1-\beta}} < u$$

Clearly condition (14) holds with equality if $u = v = 0$. It is easy to see that increases in u and v will increase the left-hand side of (14).

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