2006s-26

Non-Smooth Sustainable Development With Overshooting

Hassan Benchekroun, Seiichi Katayama, Ngo Van Long

Série Scientifique
Scientific Series

Montréal
Novembre 2006

CIRANO
Centre interuniversitaire de recherche en analyse des organisations

CIRANO

Le CIRANO est un organisme sans but lucratif constitué en vertu de la Loi des compagnies du Québec. Le financement de son infrastructure et de ses activités de recherche provient des cotisations de ses organisations-membres, d'une subvention d'infrastructure du Ministère du Développement économique et régional et de la Recherche, de même que des subventions et mandats obtenus par ses équipes de recherche.

CIRANO is a private non-profit organization incorporated under the Québec Companies Act. Its infrastructure and research activities are funded through fees paid by member organizations, an infrastructure grant from the Ministère du Développement économique et régional et de la Recherche, and grants and research mandates obtained by its research teams.

Les partenaires du CIRANO

Partenaire majeur
Ministère du Développement économique,
de l'Innovation et de l'Exportation

Partenaires corporatifs
Alcan inc.
Banque de développement du Canada
Banque du Canada
Banque Laurentienne du Canada
Banque Nationale du Canada
Banque Royale du Canada
Bell Canada
BMO Groupe financier
Bombardier
Bourse de Montréal
Caisse de dépôt et placement du Québec
Fédération des caisses Desjardins du Québec
Gaz de France
Gaz Métro
Hydro-Québec
Industrie Canada
Ministère des Finances du Québec
Pratt & Whitney Canada
Raymond Chabot Grant Thornton
Ville de Montréal

Partenaires universitaires
École Polytechnique de Montréal
HEC Montréal
McGill University
Université Concordia
Université de Montréal
Université de Sherbrooke
Université du Québec
Université du Québec à Montréal
Université Laval

Le CIRANO collabore avec de nombreux centres et chaires de recherche universitaires dont on peut consulter la liste sur son site web.

ISSN 1198-8177

Non-Smooth Sustainable Development With Overshooting

Hassan Benchekroun[*], *Seiichi Katayama*[†], *Ngo Van Long*[‡]

Résumé / *Abstract*

Nous démontrons que, dans un modèle avec la substitution entre le capital et les ressources naturelles, le sentier du développement peut être non-monotone. Si l'on commence avec un niveau faible de capital et de ressources naturelles, le sentier optimal peut dépasse le niveau du capital de l'état stationnaire. La convergence s'effectue en temps fini.

Mots clés : développement soutenable, ressources naturelles renouvelables

We show that, in a model with substitutability between capital and resources, the path of sustainable development may be non-smooth, and may exhibit the overshooting property: starting from low levels of capital and resources, the economy may accumulate capital beyond its steady-state level, before converging to it in finite time.

Keywords: *sustainable development, renewable resources*

Codes JEL : C73, H41, D60

[*] Department of Economics, McGill University, 855 Rue Sherbrooke Ouest, Montreal, Quebec, H3A 2T7, Canada. Email hassan.benchekroun@mcgill.ca

[†] Research Institute for Economics and Business Administration, Kobe University, Nada-ku, Kobe, Japan. Email: katayama@rieb.kobe-u.ac.jp

‡ Cirano and Cireq, Department of Economics, McGill University, 855 Rue Sherbrooke Ouest,, Montreal, Quebec, H3A 2T7, Canada. Email; ngo.long@mcgill.ca

1 Introduction

Since man-made capital and natural resources are substitutable inputs in the aggregate production function, a natural question that arises is how to optimally accumulate capital and manage the resource stock. The case where the natural resource stock is non-renewable has been studied by Solow under the the maximin criterion, and Dasgupta and Heal (1979) and Pezzy and Withagen (1998) under the utilitarian criterion. Solow assumed a Cobb-Douglas production function, and showed that if the share of capital is greater than the share of natural resource, then a constant path of consumption is feasible, and along such a path, the man-made capital stock increases without bound. Dasgupta and Heal (1979) and Pezzy and Withagen (1998) showed that, under the utilitarian criterion, the man-made capital stock will reach a peak, and afterwards both stocks fall to zero asymptotically. Long and Katayama (2002) obtain similar results in a differential game model of common proprty resources and private capital accumulation.

In this paper, we study the optimal path for an economy that produces an output using a stock of capital and a resource input extracted from a stock of renewable natural resource. We retain the Solow-Dasgupta-Heal assumption that capital and resource are substitutable inputs in the production of the final good, but our model differs from theirs because the resource stock is renewable. We wish to find the optimal growth path of the economy under the utilitarian criterion.We show that there exists a unique steady state with positive consumption. We ask the following questions: (i) Can it be optimal to get to the steady state in finite time under the assumption that the utility function is strictly concave? (ii) Can finite-time approach paths to the steady state be smooth, in the sense that there are no jumps in the control variables? (iii) Are there non-smooth paths to the steady state?

The answers to the above questions are as follows.

There exists a set of initial conditions (which forms a one-dimensional manifold, i.e., a curve, in the state space) such that the approach path to the steady state takes a finite time, and is smooth. If the economy starts with a low resource stock, the path along the manifold toward the steady state involves gradual accumulation of the resource stock, and gradual running down of the capital stock toward its steady state level.

If the initial conditions are not on that one-dimensional manifold, then it may be optimal to get to some point on that manifold first, and then move along the manifold to get to the steady state.The path that gets to a point on the manifold is not smooth at the time it meets the manifold.

We show that starting from low levels of capital stock and resource stock, the optimal policy consists of three phases. In phase I, the planner builds up the stock of man-made capital above its steady state level, while the resource stock is kept below its steady state level. In phase II, the capital stock declines steadily, while the resource stock continues to grow, until the steady state is reached. In phase III, the economy stays at the steady state. Thus, our model exhibits the "overshooting" property.

Before proceeding, we would like to note that there are a number of articles that are somewhat related to our paper, where the authors discussed thr optimal use patterns for renewable resources and the sustainability of economies. Clark et al.(1979) provided a general formulation with irreversible investment.They focussed on irreversibility, and did not obtain an "overshooting" result. Among the relatively recent papers, Beltratti et al.(1998) addressed the problem of optimal use of renewable resources under a variety of assumptions about the objective of that economy (with the different types of the utility function.) They constructed a model in which a man-made capital stock and a renewable resource are used for production, and give a very general characterization of the paths which are optimal in various senses. Their basic model is similar to ours, however they focused on different issues.

We are not aware of any paper which examines the precise characteristics of steady state and of the approach paths to the steady state in a model with man-made capital and renewable resource.

2 The Model

We consider a continuous-time model. Let K and S denote the stock of man-made capital, and the stock of a renewable natural resource. Let R denote the resource input. The output of the final good is

$$Y = F(K, R) = \sqrt{KR}$$

Output can be consumed, or invested. Let C denote consumption and I denote investment. Then

$$C = F(K, R) - I \tag{1}$$

Assume there is no depreciation of capital. Then

$$\dot{K} = I \tag{2}$$

Let $\theta(S)$ be the natural growth function of the resource stock. We assume it has the shape of a tent. Specifically, we assume that there exists a stock level $\widehat{S} > 0$ such that $\theta(S) = \omega S$ if $S < \widehat{S}$, and $\theta(S) = \omega\widehat{S} - \delta(S - \widehat{S})$ for $S > \widehat{S}$, where $\omega > 0$, $\delta > 0$. The net rate of growth of the resource stock is

$$\dot{S} = \theta(S) - R \tag{3}$$

Remark 1: The function $\theta(S)$ has a kink at $\widehat{S}$, so the derivative $\theta'(S)$ is not defined at $\widehat{S}$. At that point, we define the generalised gradient of $\theta(S)$, denoted by $\partial\theta$, as the real interval $[-\delta, \omega]$, where $-\delta$ is the right-hand derivative, and ω is the left-hand derivative. When applying optimal control theory, we must modify the equation for the shadow price of S when S is at $\widehat{S}$. (This will be discussed in detail later.)

The consumption C yields the utility

$$U(C) = \sqrt{C}$$

The objective of the planner is to maximize the integral of the discounted stream of utility:

$$\max \int_0^\infty \sqrt{C} e^{-\rho t} dt$$

where we assume

$$0 < \rho < \omega$$

This assumption ensures that the optimal solution involves building the resource stock to the level $\widehat{S}$.

The maximization is subject to

$$\dot{K} = \sqrt{KR} - C \tag{4}$$

$$\dot{S} = \theta(S) - R \tag{5}$$

with boundary conditions $K(0) = K_0 > 0$, $S(0) = S_0 > 0$, and

$$\lim_{t\to\infty} K(t) > 0, \ \lim_{t\to\infty} S(t) \geq 0$$

The set of positive stock levels is partitioned into two regions. Region I is the set of points (S, K) such that $0 < S < \widehat{S}$, and $K > 0$. Region II is the set of points (S, K) such that $S \geq \widehat{S}$, and $K > 0$.

We will show that there is no steady state in region I, and there is a unique steady state in region II. After that, we will show that in region I, there exists a unique one-dimensional manifold along which a smooth path converges to the steady state in region II. This manifold is downward sloping in the space (S, K), so that along the smooth convergent path, the capital stock falls and the resource stock rises. We then turn to region II and show that in that region, there exists also a unique one-dimensional manifold along which a smooth path converges to the steady state. We show that along this path, the capital stock rises and the resource stock falls.

From the above results, we infer that if the initial pair of stock levels (S_0, K_0) does not belong to either of the two manifolds, the optimal path from such an initial point, if it converges to the steady state, must either involve a jump in some control variables, or an "overshooting" along the path.

3 Necessary conditions and steady state

3.1 Necessary conditions in Region I

We define the current value Hamiltonian

$$H = \sqrt{C} + \psi_1 \left[\sqrt{KR} - C \right] + \psi_2 \left[\theta(S) - R \right]$$

where ψ_1 is the shadow price of man-made capital and ψ_2 is the shadow price of the renewable resource.

The necessary conditions are

$$\frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0 \tag{6}$$

$$\frac{\partial H}{\partial R} = \frac{1}{2}\psi_1 \sqrt{\frac{K}{R}} - \psi_2 = 0 \tag{7}$$

$$\dot{\psi}_1 = \psi_1 (\rho - \frac{1}{2}\sqrt{\frac{R}{K}}) \tag{8}$$

$$\dot{\psi}_2 = \psi_2 \left(\rho - \omega \right) \tag{9}$$

Notice that $\psi_1 > 0$ by (6). It follows that $\psi_2 > 0$ by (7). So, in region I, ψ_2 (the shadow price of the resource stock) is **always falling** because $\rho < \omega$.

Thus we obtain the following result:

Result 1: There is no steady state in Region 1.

Discussion:

Here we make some remarks about the economic meaning of the necessary conditions.

From equations (7), (8) and (9) we get

$$(\rho - \omega) - (\rho - F_K) = \frac{\dot{\psi}_2}{\psi_2} - \frac{\dot{\psi}_1}{\psi_1} = \frac{1}{F_R}\frac{d(F_R)}{dt}$$

Hence

$$F_K = \omega + \frac{1}{F_R}\frac{d(F_R)}{dt} \tag{10a}$$

We may call equation (10a) the *Modified Hotelling Rule:* the rate of capital gain (rate of increase in the price of the extracted resource) plus the biological growth rate must be equated to the rate of interest on the capital good, F_K.

From (6) and (8), we get

$$\frac{\dot{C}}{2C} = F_K - \rho \tag{11}$$

which is the *Ramsey-Euler Rule:* the proportional rate of consumption growth, multiplied by the elasticity of marginal utility, must be equated to the difference between the rate of interest F_K and the utility-discount rate, ρ.

It is convenient to define a new variable x :

$$x(t) = \frac{K(t)}{R(t)}$$

This variable is the capital/resource-input ratio, and is a measure of the *capital intensity* of the production process at time t.

Using (7) we get

$$x(t) = \left[\frac{2\psi_2(t)}{\psi_1(t)}\right]^2 \tag{12}$$

From this equation, we get

Result 2: $x(t)$ jumps at some time t_1 only if either ψ_1 or ψ_2 jumps at t_1.

Discussion: ψ_2 is continuous in Region I, but when $S(t)$ reaches $\widehat{S}$ (which is in Region II) the kink in the growth function $\theta(S)$ may cause ψ_2 to jump.

3.2 The necessary conditions in Region II

The necessary conditions for Region II are a bit more complicated, because at the point $\widehat{S}$ the function $\theta(S)$ is not differentiable. Thus we must deal with a "non-smooth" problem. For a general treatment of non-smooth optimal control problem see Clarke and Winter (1983), or Clarke (1983); here we follow the exposition in Docker et al (2000, pages 74-79).

Since $\theta(S)$ has a kink at $\widehat{S}$, with left-hand derivative equal to $\omega > 0$ and right-hand derivative equal $-\delta$, the generalized gradient of $\theta(.)$ at $\widehat{S}$ is defined as

$$\partial\theta(\widehat{S}) = [-\delta, \omega]$$

The necessary conditions are

$$\frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0 \tag{13}$$

$$\frac{\partial H}{\partial R} = \frac{1}{2}\psi_1\sqrt{\frac{K}{R}} - \psi_2 = 0 = 0 \tag{14}$$

$$\dot{K} = \sqrt{KR} - C \tag{15}$$

$$\dot{S} = \omega\widehat{S} - \delta(S - \widehat{S}) - R \text{ if } S > \widehat{S} \tag{16}$$

$$\dot{\psi}_1 = \psi_1(\rho - \frac{1}{2}\sqrt{\frac{R}{K}}) \tag{17}$$

and, from Docker et al. (2000, pages 74-79),

$$-(\dot{\psi}_2 - \rho\psi_2) \in [-\delta\psi_2, \omega\psi_2] \text{ if } S = \widehat{S} \tag{18}$$

$$-(\dot{\psi}_2 - \rho\psi_2) = -\delta\psi_2 \text{ if } S > \widehat{S} \tag{19}$$

Result 3 : There exists a unique steady state in Region II. The steady state resource stock is

$$S_{ss} = \widehat{S}$$

and the steady state capital stock is

$$K_{ss} = K_{ss} = \omega\widehat{S}\left[\frac{\alpha}{\rho}\right]^{1/(1-\alpha)}$$

Proof:

Let us find the corresponding steady state values of other variables. From (16), at the steady state,

$$R_{ss} = \omega\widehat{S}. \tag{20}$$

From (17), at the steady state,

$$\alpha\left(\frac{K}{R}\right)^{\alpha-1} = \rho \tag{21}$$

Thus

$$K_{ss} = \omega\widehat{S}\left[\frac{\alpha}{\rho}\right]^{1/(1-\alpha)} \tag{22}$$

$$x_{ss} = \frac{K_{ss}}{R_{ss}} = \left[\frac{\alpha}{\rho}\right]^{1/(1-\alpha)} \tag{23}$$

Using (15), at the steady state

$$C_{ss} = \omega\widehat{S}\left[\frac{\alpha}{\rho}\right]^{\alpha/(1-\alpha)} \tag{24}$$

Thus, from (13) and (24)

$$\psi_{ss1} = \left(\omega\widehat{S}\right)^{-\gamma}\left[\frac{\alpha}{\rho}\right]^{-\alpha\gamma/(1-\alpha)}$$

and, from (14)

$$\psi_{ss2} = \left(\omega \widehat{S}\right)^{-\gamma} (1-\alpha) \left[\frac{\alpha}{\rho}\right]^{\alpha(1-\gamma)/(1-\alpha)}$$

which is consistent with (18) because $\rho \in [-\delta, \omega]$.

4 Dynamics in Region I

Since the steady state in region II is at the the boundary between the two regions, we are particularly interested in paths in Region I that converges to the steady state in region II, i.e. $(S(t), K(t)) \to (S_{ss}, K_{ss})$ in finite or infinite time. An important subclass of such convergent paths is called the paths of smooth convergent paths, by which we mean the control variables $C(t)$ and $R(t)$ do not jump (and hence $x(t)$ does not jump).

4.1 The time path of capital/resource-input ratio in Region I

Lemma 1: In region I, the time path of the capital/resource-input ratio, $x(t)$, satisfies the differential equation:

$$-\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} = -\omega \tag{25}$$

It follows that:

(1) if x_0 is optimally chosen, then

$$x(t) = \left(\left(\sqrt{x_0} - \frac{1}{2\omega}\right)e^{-\omega t} + \frac{1}{2\omega}\right)^2$$

(2) if at some time T, the variable x takes the value x_T, then

$$x(t) = \left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right)e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2 \equiv [g(t; x_T, T)]^2 \tag{26}$$

Proof: See Appendix A1.

Remark 2: If we impose the condition that at some time T the variable $x(T)$ takes the following value (which is its steady state value in region II)

$$x_T = (\frac{1}{2\rho})^2 = x_{ss} \tag{27}$$

then we can say something more definite about $x(t)$. See Lemma 2 below.

Lemma 2: If $x(t) \to x_T = x_{ss}$, then over the time interval $[0, T]$ the capital/resource-input ratio $x(t)$ **decreases steadily**.

Proof: From (26)

$$\dot{x}(t) = 2g(t; x_T, T)g'(t; x_T, T) = -2\omega \left(\sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\omega(t-T)} g(t; x_T, T) < 0$$

because

$$\sqrt{x_T} = \frac{1}{2\rho} > \frac{1}{2\omega}$$

Remark 3: It can be shown (see Appendix A2) that if $x_T = x_{ss}$ then

$$\frac{\dot{x}(t)}{x(t)} = -2\omega \left(1 - \frac{1}{\left(\left(\frac{\omega}{\rho} - 1 \right) e^{-\omega(t-T)} + 1 \right)} \right)$$

4.2 The time path of ψ_1

Lemma 3: In region I, the time path of ψ_1 is

$$\psi_1(t) = \psi_{1T} \frac{\sqrt{x_T}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega} \right) e^{-\rho(t-T)} + \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)} \right)}$$

Proof: See Appendix A3.

Lemma 4: if $x(t) \to x_T = x_{ss}$ then over the time interval $[0, T]$,the shadow price of capital, $\psi_1(t)$, **increases steadily**.

Proof: See Appendix A4.

4.3 The time path of consumption in Region I

Lemma 5: In region I, the time path of consumption is

$$C(t) = C_T \frac{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2}{x_T} e^{-2(\rho-\omega)(t-T)} = C_T \left[\frac{x(t)}{x_T}\right] e^{-2(\rho-\omega)(t-T)}$$

If $x(t) \to x_T = x_{ss}$ over the time interval $[0, T]$, then consumption **decreases** steadily.

Proof: See Appendix A5.

4.4 The time path of extraction in Region I

Lemma 6: In region I, the time path of extraction is

$$R(t) = \left(\frac{1}{2\rho}\right) \frac{C_T}{x_T} \exp\left(2\omega t - 2\omega T - 2\rho t + 2\rho T\right) + e^{2\omega t} E$$

where E satisfies

$$R_T = \left(\frac{1}{2\rho}\right) \frac{C_T}{x_T} + e^{2\omega T} E$$

Thus, **if** (i) $x(t) \to x_T = x_{ss}$, (ii) $C(t) \to C_T = C_{ss}$ and (iii) $R(t) \to R_T = R_{ss}$ then $E = 0$, and extraction will be rising steadily:

$$\dot{R}(t) = 2(\omega - \rho)\,\omega \widehat{S} e^{2(\omega-\rho)(t-T)} > 0$$

Proof: See Appendix 6.

4.5 The path of capital in region I

We now turn to the capital, we have

Lemma 7: Along the optimal path in Region I

$$\frac{\dot{K}}{K} = x^{-\frac{1}{2}} - 2\rho$$

Along a smooth convergent path (i.e. $x(t) \to x_T = x_{ss}$) the capital stock $K(t)$ falls steadily.

Proof: See Appendix A7.

Lemma 8: Along a smooth convergent path in Region I, there is a positive relationship between the time T and the initial stock K_0. It is given by

$$K_0 = K(0) = \omega \widehat{S} e^{-2(\omega-\rho)T} \left(\left(\frac{1}{2\rho} - \frac{1}{2\omega} \right) e^{\omega T} + \frac{1}{2\omega} \right)^2 \tag{28}$$

with

$$\frac{dK_0}{dT} > 0 \tag{29}$$

Proof: See Appendix A8.

4.6 The path of the resource stock in Region I

In Region I, the resource stock follows the law of motion

$$\dot{S} = \omega S - R$$

Thus we get

Lemma 9: Along a smooth convergent path in Region I (with $R(t) \to R_T = R_{ss} = \omega \widehat{S}$) , there is a negative relationship between the time T and the initial resource stock S_0.

$$S_0 = e^{-\omega T} \widehat{S} \left(\omega \frac{\left(e^{-(\omega-2\rho)T} - 1\right)}{-\omega + 2\rho} + 1 \right) \tag{30}$$

with

$$\frac{dS_0}{dT} < 0 \tag{31}$$

Proof: See Appendix 9.

PROPOSITION 1: In Region I, the set of initial stock pairs (S, K) from which the optimal path is a smooth convergent path is the one dimensional manifold defined by the two equations (28) and (30). This manifold has a negative slope in the space (S, K).

Proof: Use (29) and (31):

$$\frac{dK_0}{dS_0} < 0$$

5 Dynamics in Region II

5.1 The time path of capital/resource-input ratio in Region II:

Lemma 1b: In region II, the time path of the capital/resource-input ratio, $x(t)$, satisfies the differential equation:

$$-\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} = \delta \tag{32}$$

It follows that:

(1) if x_0 is optimally chosen, then

$$x(t) = x(t) = \left(\left(\sqrt{x_0} + \frac{1}{2\delta}\right)e^{\delta t} - \frac{1}{2\delta}\right)^2$$

(2) if at some time T, the variable x takes the value x_T, then

$$x(t) = \left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{\delta(t-T)} - \frac{1}{2\delta}\right)^2 \equiv [f(t; x_T, T)]^2$$

Proof: See Appendix 10

Lemma 2: In Region II, over the time interval $[0, T]$ the capital/resource-input ratio $x(t)$ **increases steadily**.

Proof: from (32)

$$-\frac{1}{2} + \frac{1}{2}\frac{\dot{x}}{\sqrt{x}} = \delta x^{\frac{1}{2}}$$

so $\dot{x}$ must be positive.

Remark 3b: We retrieve the results of Region I if we substitute δ by $-\omega$

5.2 The path of ψ_1 in region II

Lemma 3b: In region I, the time path of ψ_1 is

$$\psi_1(t) = \frac{\psi_1(T)\, e^{(\delta+\rho)(t-T)}}{\left(\left(1+\frac{\rho}{\delta}\right) e^{\delta(t-T)} - \frac{\rho}{\delta}\right)}$$

Proof: See Appendix 11.

5.3 The time path of consumption in Region II

Lemma 5b: In region II, the time path of consumption is

$$c(t) = C_T \left(\frac{\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)} - \frac{1}{2\delta}\right)}{\sqrt{x_T}} e^{-(\delta+\rho)(t-T)} \right)^2 = C_T \left[\frac{x(t)}{x_T} \right] e^{-2(\rho+\delta)(t-T)}$$

If $x(t) \to x_T = x_{ss}$ over the time interval $[0, T]$, then consumption **decreases** steadily.

Proof: See Appendix 12.

5.4 The path of extraction in region II

Lemma 6b: In region I, the time path of extraction satisfies the differential equation

$$\dot{R} = -2\delta R - \frac{C_T e^{-2(\delta+\rho)(t-T)}}{x_T}$$

Thus, **if** (i) $x(t) \to x_T = x_{ss}$, (ii) $C(t) \to C_T = C_{ss}$ and (iii) $R(t) \to R_T = R_{ss}$

$$R(t) = \omega \widehat{S} e^{-2(\delta+\rho)(t-T)}$$

$$\dot{R}(t) < 0$$

Proof: See Appendix 13

5.5 The path of capital in region II

We now turn to the capital, we have

Lemma 7b: Along the optimal path in Region II

$$\frac{\dot{K}}{K} = x^{-\frac{1}{2}} - 2\rho$$

Along a smooth convergent path (i.e. $x(t) \rightarrow x_T = x_{ss}$) the capital stock $K(t)$ falls steadily.

Proof: See Appendix 14.

Lemma 8b: Along a smooth convergent path in Region II, there is a positive relationship between the time T and the initial stock K_0. It is given by

$$K_0 = K(0) = \omega\widehat{S}\left(\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{-\delta T} - \frac{1}{2\delta}\right)e^{(\delta+\rho)T}\right)^2$$

Proof: See Appendix 15.

5.6 The path of the resource stock in Region II

In region II we have

$$\dot{S} = \omega\widehat{S} - \delta(S - \widehat{S}) - R$$

Substituting R gives

$$\dot{S} = \omega\widehat{S} - \delta(S - \widehat{S}) - \omega\widehat{S}e^{-2(\delta+\rho)(t-T)}$$

$$\dot{S} + \delta S = \omega\widehat{S} + \delta\widehat{S} - \omega\widehat{S}e^{-2(\delta+\rho)(t-T)}$$

$$\dot{S} + \delta S = \widehat{S}\left(\omega + \delta - \omega e^{-2(\delta+\rho)(t-T)}\right)$$

Note that

$$\dot{S}(T) + \delta\widehat{S} = \delta\widehat{S}$$

and thus

$$\dot{S}(T) = 0$$

We now solve for the path of the resource stock

$$\begin{aligned} S' &= A\left(\omega + \delta - \omega e^{-2(\delta+\rho)(t-T)}\right) - \delta S \\ S(T) &= A \end{aligned}$$

The exact solution is:

$$S(t) = \omega\frac{A}{\delta} + A - \omega\frac{A}{-\delta - 2\rho}\exp(-2\delta t + 2\delta T - 2\rho t + 2\rho T) - 2e^{-\delta t}\omega A\frac{\delta + \rho}{e^{-\delta T}\delta(\delta + 2\rho)}$$

$$S(t) = \omega\frac{\widehat{S}}{\delta} + \widehat{S} + \omega\frac{\widehat{S}}{\delta + 2\rho}e^{-2(\delta+\rho)(t-T)} - 2e^{-\delta(t-T)}\omega\widehat{S}\frac{\delta + \rho}{\delta(\delta + 2\rho)}$$

we check that $S(T) = \omega\frac{\widehat{S}}{\delta} + \widehat{S} + \omega\frac{\widehat{S}}{\delta+2\rho} - 2\omega\widehat{S}\frac{\delta+\rho}{\delta(\delta+2\rho)} = \widehat{S}$

Moreover

$$\dot{S}(t) = \omega\widehat{S}\left(-\frac{2(\delta + \rho)}{\delta + 2\rho}e^{-2(\delta+\rho)(t-T)} + 2\delta e^{-\delta(t-T)}\frac{\delta + \rho}{\delta(\delta + 2\rho)}\right)$$

$$\dot{S}(t) = 2\omega\widehat{S}2e^{-\delta(t-T)}\frac{\delta + \rho}{(\delta + 2\rho)}\left(-e^{-(\delta+2\rho)(t-T)} + 1\right) < 0$$

There exists a smooth path reaching $\widehat{S}$ at T if S_0 satisfies

$$S_0 = S(0) = \omega\frac{\widehat{S}}{\delta} + \widehat{S} + \omega\frac{\widehat{S}}{\delta + 2\rho}e^{2(\delta+\rho)T} - 2e^{\delta T}\omega\widehat{S}\frac{\delta + \rho}{\delta(\delta + 2\rho)}$$

$$\frac{dS_0}{dT} = \omega\frac{2(\delta + \rho)\widehat{S}}{\delta + 2\rho}e^{2(\delta+\rho)T} - 2e^{\delta T}\omega\widehat{S}\frac{\delta + \rho}{(\delta + 2\rho)}$$

$$\frac{dS_0}{dT} = 2e^{\delta T}\omega\widehat{S}\frac{\delta + \rho}{(\delta + 2\rho)}\left(e^{(\delta+2\rho)T} - 1\right) > 0$$

In Region II we also have

$$\frac{dS_0}{dT} > 0$$

and

$$\frac{dK_0}{dT} < 0$$

so

$$\frac{dK_0}{dS_0} < 0$$

In Both region I and region II we have

$$\frac{dK_0}{dS_0} < 0$$

This implies either overshooting or jump in the control paths.

6 Concluding Remarks

We have been able to show that the path to a steady state may exhibit the overshooting property. The economy accumulate capital to some level much higher than its steady-state level, before running it down. This is because when the renewable resource is still at a low level, more output can be generated by accumulating capital, while using the resource sparingly. When a sufficient large level of resource has been achieved, it becomes more efficient to use more resource, and less capital, in the production process.

Our model displays two additional features: it takes a finite time to get to the steady state, and the paths to the steady state is generally non-smooth, unless the economy happens to have a combination of stock levels that lies on the smooth one-dimensional manifold.

Acknowledgments: We thank Richard Hartl, Kim Long, and Akio Matsumoto for comments, and SSHRC and FQRSC for financial support.

APPENDIX 1:

Proof of Lemma 1

Step 1:

We first show that x satisfies the following differential equation This is shown from the necessary conditions (**??**), (6) and (8),

$$\frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0 \tag{33}$$

$$\frac{1}{2\sqrt{C}} = \psi_1 \tag{34}$$

$$-\frac{1}{2}\ln C = \ln \psi_1 + \ln 2 \tag{35}$$

$$-\frac{\dot{C}}{2C} = \frac{\dot{\psi}_1}{\psi_1} \tag{36}$$

but

$$\frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{2}x^{-\frac{1}{2}} \tag{37}$$

so we have

$$-\frac{\dot{C}}{2C} = \frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{2}x^{-\frac{1}{2}} \tag{38}$$

We get the relationship between ψ_1 and x

$$\frac{\partial H}{\partial R} = \frac{1}{2}\psi_1\sqrt{\frac{K}{R}} - \psi_2 = 0 \tag{39}$$

or

$$\frac{1}{2}\psi_1\sqrt{x} - \psi_2 = 0 \tag{40}$$

$$-\ln 2 + \ln \psi_1 + \frac{1}{2}\ln x - \psi_2 = 0 \tag{41}$$

so that

$$\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2}\frac{\dot{x}}{x} \tag{42}$$

and

$$(\rho - \omega) = \rho - \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} \tag{43}$$

or

$$-\omega = -\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} \tag{44}$$

This ends Step 1.

Step 2: Solving for $x(t)$:

We have

$$-\omega = -\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} \tag{45}$$

multiplying each side by $\sqrt{x}$ gives

$$-\frac{1}{2} + \frac{1}{2}\frac{\dot{x}}{\sqrt{x}} = -\omega x^{\frac{1}{2}} \tag{46}$$

let $y \equiv \sqrt{x}$

$$-\frac{1}{2} + \dot{y} = -\omega y \tag{47}$$

the solution can be written in two forms:

$$y(t) = \left(y_0 - \frac{1}{2}\right)e^{-\omega t} + \frac{1}{2\omega}$$

or

$$y(t) = \left(y_T - \frac{1}{2\omega}\right)e^{-\omega(t-T)} + \frac{1}{2\omega}$$

where $y_0 = y(0)$ or $y_T = y(T)$ and therefore we have

$$x(t) = \left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right)e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2 \tag{48}$$

APPENDIX 2:

If $x_T = x_{ss}$ then

$$\frac{\dot{x}(t)}{x(t)} = -2\omega\frac{\left(\sqrt{x_T} - \frac{1}{2\omega}\right)e^{-\omega(t-T)}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right)e^{-\omega(t-T)} + \frac{1}{2\omega}\right)}$$

$$\frac{\dot{x}(t)}{x(t)} = -2\omega\left(1 - \frac{1/2\omega}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right)e^{-\omega(t-T)} + \frac{1}{2\omega}\right)}\right)$$

$$\frac{\dot{x}(t)}{x(t)} = -2\omega \left(1 - \frac{1/2\omega}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)}\right)$$

$$\frac{\dot{x}(t)}{x(t)} = -2\omega \left(1 - \frac{1/2\omega}{\left(\left(\frac{1}{2\rho} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)}\right)$$

$$\frac{\dot{x}(t)}{x(t)} = -2\omega \left(1 - \frac{1}{\left(\left(\frac{\omega}{\rho} - 1\right) e^{-\omega(t-T)} + 1\right)}\right)$$

so since $\left(\frac{\omega}{\rho} - 1\right) > 0$ then $\frac{\dot{x}(t)}{x(t)} < 0$.

APPENDIX 3: Proof of Lemma 3.

We can solve for ψ_1 from (12)

$$\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2}\frac{\dot{x}}{x} \tag{49}$$

with

$$\dot{\psi}_2 = \psi_2 (\rho - \omega) \tag{50}$$

so

$$\frac{\dot{\psi}_1}{\psi_1} = \rho - \omega - \frac{1}{2}\frac{\dot{x}}{x}$$

integrating gives

$$\ln \frac{\psi_1(t)}{\psi_1(T)} = (\rho - \omega)(t - T) - \ln \sqrt{\frac{x(t)}{x(T)}}$$

or

$$\psi_1(t) = \psi_{1T} \frac{\sqrt{x_T}}{\sqrt{x(t)}} e^{(\rho-\omega)(t-T)}$$

$$\psi_1(t) = \psi_{1T} \frac{\sqrt{x_T}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)} e^{(\rho-\omega)(t-T)}$$

$$\psi_1(t) = \psi_{1T} \frac{\sqrt{x_T}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\rho(t-T)} + \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}\right)}$$

APPENDIX 4: Proof of Lemma 3

The denominator is $D(t) = \left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\rho(t-T)} + \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}$ is such that

$$D'(t) = -\rho \left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\rho(t-T)} - (\rho - \omega) \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}$$

$$D'(t) = -\rho \left(\frac{1}{2\rho} - \frac{1}{2\omega}\right) e^{-\rho(t-T)} - (\rho - \omega) \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}$$

$$D'(t) = \frac{1}{2\omega} \left(-\rho \left(\frac{\omega}{\rho} - 1\right) e^{-\rho(t-T)} - (\rho - \omega) e^{-(\rho-\omega)(t-T)}\right)$$

$$D'(t) = \frac{1}{2\omega} \left((\rho - \omega) e^{-\rho(t-T)} - (\rho - \omega) e^{-(\rho-\omega)(t-T)}\right)$$

$$D'(t) = \frac{1}{2\omega} (\rho - \omega) e^{-\rho(t-T)} \left(1 - e^{\omega(t-T)}\right) < 0$$

since $\rho < \omega$. So

$$\dot{\psi}_1(t) > 0$$

APPENDIX 5: Proof of Lemma 5

$$\frac{1}{2\sqrt{C}} - \psi_1 = 0$$

or

$$\frac{1}{2\psi_1} = \sqrt{C}$$

or

$$\left(\frac{1}{2\psi_1}\right)^2 = C$$

that is

$$C(t) = \frac{1}{\left(2\psi_{1T} \frac{\sqrt{x_T}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)} e^{(\rho-\omega)(t-T)}\right)^2}$$

$$C(t) = \frac{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2}{x_T (2\psi_{1T})^2} e^{-2(\rho-\omega)(t-T)}$$

$$C(t) = C_T \frac{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2}{x_T} e^{-2(\rho-\omega)(t-T)} = c_T \left[\frac{x(t)}{x_T}\right] e^{-2(\rho-\omega)(t-T)}$$

The evolution of the consumption path is given by

$$\frac{\dot{C}}{C} = -2\frac{\dot{\psi}_1}{\psi_1}$$

If $x(t) \to x_T = x_{ss}$ over the time interval $[0, T]$ the C falls steadily because ψ_1 rises steadily.

APPENDIX 6

From the definition of $x = K/R$, we have

$$\dot{K} = \dot{R}x + R\dot{x}$$

and

$$\dot{K} = \sqrt{KR} - C = R\sqrt{x} - C$$

so

$$\dot{R}x + R\dot{x} = R\sqrt{x} - C \tag{51}$$

$$\dot{R}x = R\left(\sqrt{x} - \dot{x}\right) - C \tag{52}$$

or

$$\dot{R} = R\left(\frac{1}{\sqrt{x}} - \frac{\dot{x}}{x}\right) - \frac{C}{x} \tag{53}$$

using (25) yields

$$-\omega = -\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x}$$

so

$$\dot{R} = 2\omega R - \frac{C}{x} \tag{54}$$

where

$$C(t) = C_T \frac{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2}{x_T} e^{-2(\rho-\omega)(t-T)}$$

and

$$x(t) = \left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2$$

so

$$\dot{R} = 2\omega R - \frac{C_T \frac{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2}{x_T} e^{-2(\rho-\omega)(t-T)}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2} \tag{55}$$

Hence

$$\dot{R} = 2\omega R - \frac{C_T e^{-2(\rho-\omega)(t-T)}}{x_T} \tag{56}$$

The solution is

$$R(t) = \frac{1}{2} \frac{C_T}{x_T \rho} \exp\left(2\omega t - 2\omega T - 2\rho t + 2\rho T\right) + e^{2\omega t} E$$

and

$$R_T = \frac{C_T}{x_T 2\rho} + e^{2\omega T} E$$

with

$$x_T = \frac{K_{ss}}{R_{ss}} = \left[\frac{1}{2\rho}\right]^2$$

$$C_T = \omega \widehat{S} \left[\frac{1}{2\rho}\right] \tag{57}$$

Now

$$R_T = \omega \widehat{S} \tag{58}$$

so

$$E = \left(R_T - \frac{C_T}{x_T 2\rho}\right) e^{-2\omega T} = 0$$

and

$$R(t) = \omega \widehat{S} e^{2(\omega-\rho)(t-T)} \tag{59}$$

$$\dot{R}(t) = 2(\omega - \rho)\, \omega \widehat{S} e^{2(\omega-\rho)(t-T)} > 0$$

APPENDIX 7: Proof of Lemma 7

$$K = xR$$

$$\frac{\dot{K}}{K} = \frac{\dot{x}}{x} + \frac{\dot{R}}{R} = \frac{\dot{x}}{x} + 2\left(\omega - \rho\right)$$
$$\frac{\dot{K}}{K} = -2\omega + x^{-\frac{1}{2}} + 2\left(\omega - \rho\right)$$
$$\frac{\dot{K}}{K} = x^{-\frac{1}{2}} - 2\rho$$

since $\dot{x} < 0$ we have $\frac{dx^{-\frac{1}{2}}}{dt} > 0$ with $\left(x\left(T\right)\right)^{-\frac{1}{2}} = 2$ and therefore $x^{-\frac{1}{2}} - 2\rho < 0$ for all $t < T$ and thus

$$\frac{\dot{K}}{K} = x^{-\frac{1}{2}} - 2\rho < 0$$

APPENDIX 8: Proof of Lemma 8

Substituting for x and R gives

$$K = xR = \omega\widehat{S}e^{2(\omega-\rho)(t-T)}\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right)e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2$$

at time $t = 0$ we have

$$K_0 = K\left(0\right) = \omega\widehat{S}e^{-2(\omega-\rho)T}\left(\left(\frac{1}{2\rho} - \frac{1}{2\omega}\right)e^{\omega T} + \frac{1}{2\omega}\right)^2$$

$$\frac{dK_0}{dT} = \omega\widehat{S}e^{-2(\omega-\rho)T}\left(\left(\frac{1}{2\rho} - \frac{1}{2\omega}\right)e^{\omega T} + \frac{1}{2\omega}\right)^2$$

$$\frac{dK_0}{dT} = \omega\widehat{S}\frac{d\left(\left(e^{-(\omega-\rho)T}\left(\left(\frac{1}{2\rho} - \frac{1}{2\omega}\right)e^{\omega T} + \frac{1}{2\omega}\right)\right)^2\right)}{dT}$$

Let $f\left(T\right) = e^{-(\omega-\rho)T}\left(\left(\frac{1}{2\rho} - \frac{1}{2\omega}\right)e^{\omega T} + \frac{1}{2\omega}\right)$ we have

$$f\left(T\right) = \left(\frac{1}{2\rho} - \frac{1}{2\omega}\right)e^{\rho T} + \frac{1}{2\omega}e^{-(\omega-\rho)T}$$

$$f'\left(T\right) = \rho\left(\frac{1}{2\rho} - \frac{1}{2\omega}\right)e^{\rho T} - \left(\omega - \rho\right)\frac{1}{2\omega}e^{-(\omega-\rho)T}$$

$$f'\left(T\right) = \frac{1}{2\omega}\left(\rho\left(\frac{\omega}{\rho} - 1\right)e^{\rho T} - \left(\omega - \rho\right)e^{-(\omega-\rho)T}\right)$$

$$f'(T) = \frac{1}{2\omega}(\omega - \rho)\, e^{\rho T}\left(1 - e^{-\omega T}\right) > 0$$

So

$$\frac{dK_0}{dT} = \omega \widehat{S} 2 f'(T)\, f(T) > 0$$

APPENDIX 9: Proof of Lemma 9.

Substituting R to get

$$\dot{S} = \omega S - \omega \widehat{S} e^{2(\omega-\rho)(t-T)}$$

The exact solution is:

$$S(t) = \widehat{S}\left(\frac{\omega}{-\omega+2\rho} e^{2(\omega-\rho)(t-T)} + 2e^{\omega(t-T)}\frac{-\omega+\rho}{(-\omega+2\rho)}\right)$$

$$S(t) = \widehat{S}\left[\frac{2(\omega-\rho)\, e^{-\omega(T-t)} - \omega e^{-2(\omega-\rho)(T-t)}}{(2(\omega-\rho)-\omega)}\right]$$

$$\dot{S}(t) = \widehat{S}\left(\frac{\omega 2(\omega-\rho)}{-\omega+2\rho} e^{2(\omega-\rho)(t-T)} + 2\omega e^{\omega(t-T)}\frac{-\omega+\rho}{(-\omega+2\rho)}\right)$$

$$\dot{S}(t) = 2\omega(\omega-\rho)\, e^{\omega(t-T)}\widehat{S}\left(\frac{1}{-\omega+2\rho} e^{(\omega-2\rho)(t-T)} - \frac{1}{(-\omega+2\rho)}\right)$$

$$\dot{S}(t) = 2\omega(\omega-\rho)\, e^{\omega(t-T)}\widehat{S}\left(\frac{e^{(\omega-2\rho)(t-T)} - 1}{(-\omega+2\rho)}\right) > 0$$

The initial stock must be

$$S_0 = S(0) = \widehat{S}\left(\frac{\omega}{-\omega+2\rho} e^{-2(\omega-\rho)T} + 2e^{-\omega T}\frac{-\omega+\rho}{(-\omega+2\rho)}\right)$$

$$S_0 = e^{-\omega T}\widehat{S}\left(\omega\frac{\left(e^{-(\omega-2\rho)T} - 1\right)}{-\omega+2\rho} + 1\right)$$

$$\frac{dS_0}{dT} = \widehat{S}\left(\frac{-2(\omega-\rho)\,\omega}{-\omega+2\rho} e^{-2(\omega-\rho)T} - \omega 2e^{-\omega T}\frac{-\omega+\rho}{(-\omega+2\rho)}\right)$$

$$\frac{dS_0}{dT} = 2(\omega-\rho)\,\omega\widehat{S} e^{-\omega T}\left(\frac{1 - e^{-(\omega-2\rho)T}}{-\omega+2\rho}\right) < 0$$

So we have

$$\frac{dS_0}{dT} = 2\left(\omega - \rho\right)\omega\widehat{S}e^{-\omega T}\left(\frac{1 - e^{-(\omega - 2\rho)T}}{-\omega + 2\rho}\right) < 0$$

and

$$\frac{dK_0}{dT} > 0$$

and therefore in Region I:

$$\frac{dK_0}{dS_0} < 0$$

APPENDIX 10. Proof of Lemma 1b.

Step 1:

We first show that x satisfies the following differential equation

$$-\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} = \delta \tag{60}$$

This is shown from the necessary conditions (**??**), (6) and (8),

$$\frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0 \tag{61}$$

$$\frac{1}{2\sqrt{C}} = \psi_1 \tag{62}$$

$$-\frac{1}{2}\ln C = \ln\psi_1 + \ln 2 \tag{63}$$

$$-\frac{\dot{C}}{2C} = \frac{\dot{\psi}_1}{\psi_1} \tag{64}$$

but we know from the necessary conditions

$$\frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{2}x^{-\frac{1}{2}} \tag{65}$$

so we have

$$-\frac{\dot{C}}{2C} = \frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{2}x^{-\frac{1}{2}} \tag{66}$$

The relationship between ψ_1 and x is from

$$\frac{\partial H}{\partial R} = \frac{1}{2}\psi_1\sqrt{\frac{K}{R}} - \psi_2 = 0 \tag{67}$$

or

$$\frac{1}{2}\psi_1\sqrt{x} - \psi_2 = 0 \tag{68}$$

$$-\ln 2 + \ln\psi_1 + \frac{1}{2}\ln x - \psi_2 = 0 \tag{69}$$

so that

$$\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2}\frac{\dot{x}}{x} \tag{70}$$

and from the necessary conditions we have

$$\dot{\psi}_2 = \psi_2(\delta + \rho) \tag{71}$$

so we have

$$\delta + \rho = \rho - \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} \tag{72}$$

or

$$\delta = -\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} \tag{73}$$

This ends Step 1

Step 2: Solving for $x\,(t)$:

We have

$$\delta = -\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} \tag{74}$$

multiplying each side by $\sqrt{x}$ gives

$$-\frac{1}{2} + \frac{1}{2}\frac{\dot{x}}{\sqrt{x}} = \delta x^{\frac{1}{2}} \tag{75}$$

let $y \equiv \sqrt{x}$

$$-\frac{1}{2} + \dot{y} = \delta y \tag{76}$$

the solution can be written in two forms:

$$y(t) = \left(y_0 + \frac{1}{2\delta}\right) e^{\delta t} - \frac{1}{2\delta}$$

or

$$y(t) = \left(y_T + \frac{1}{2\delta}\right) e^{\delta(t-T)} - \frac{1}{2\delta}$$

where $y_0 = y(0)$ or $y_T = y(T)$ and therefore we have

$$x(t) = \left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)} - \frac{1}{2\delta}\right)^2$$

or

$$x(t) = \left(\left(\sqrt{x_0} + \frac{1}{2\delta}\right) e^{\delta t} - \frac{1}{2\delta}\right)^2$$

This ends Step 2.

APPENDIX 11: Proof of Lemma 3b

We have

$$\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2}\frac{\dot{x}}{x}$$

$$\delta + \rho = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2}\frac{\dot{x}}{x} \tag{77}$$

$$\delta + \rho = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2}\frac{2\delta\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)}\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)} - \frac{1}{2\delta}\right)}{\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)} - \frac{1}{2\delta}\right)^2} \tag{78}$$

$$\delta + \rho = \frac{\dot{\psi}_1}{\psi_1} + \frac{\delta\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)}}{\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)} - \frac{1}{2\delta}\right)} \tag{79}$$

$$(\delta + \rho)(t - T) = \ln\frac{\psi_1(t)}{\psi_1(T)} + \ln\frac{\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)} - \frac{1}{2\delta}\right)}{\sqrt{x_T}} \tag{80}$$

$$e^{(\delta+\rho)(t-T)} = \frac{\psi_1(t)}{\psi_1(T)}\frac{\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)} - \frac{1}{2\delta}\right)}{\sqrt{x_T}} \tag{81}$$

$$\frac{\sqrt{x_T}\psi_1(T)\, e^{(\delta+\rho)(t-T)}}{\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{\delta(t-T)} - \frac{1}{2\delta}\right)} = \psi_1(t) \tag{82}$$

$$\frac{\frac{1}{2\rho}\psi_1(T)\,e^{(\delta+\rho)(t-T)}}{\left(\left(\frac{1}{2\rho}+\frac{1}{2\delta}\right)e^{\delta(t-T)}-\frac{1}{2\delta}\right)}=\psi_1(t) \tag{83}$$

$$\frac{\psi_1(T)\,e^{(\delta+\rho)(t-T)}}{\left(\left(1+\frac{\rho}{\delta}\right)e^{\delta(t-T)}-\frac{\rho}{\delta}\right)}=\psi_1(t) \tag{84}$$

APPENDIX 12

From

$$\frac{1}{2\sqrt{C}}=\psi_1 \tag{85}$$

we have

$$C=\left(\frac{1}{2\psi_1}\right)^2$$

$$C=\left(\frac{\left(\left(\sqrt{x_T}+\frac{1}{2\delta}\right)e^{\delta(t-T)}-\frac{1}{2\delta}\right)}{2\sqrt{x_T}\psi_1(T)\,e^{(\delta+\rho)(t-T)}}\right)^2$$

$$C=C_T\left(\frac{\left(\left(\sqrt{x_T}+\frac{1}{2\delta}\right)e^{\delta(t-T)}-\frac{1}{2\delta}\right)}{\sqrt{x_T}}e^{-(\delta+\rho)(t-T)}\right)^2$$

$$C=C_T\left(\left(\left(1+\frac{\rho}{\delta}\right)e^{\delta(t-T)}-\frac{\rho}{\delta}\right)e^{-(\delta+\rho)(t-T)}\right)^2$$

$$C=C_T\left(\left(1+\frac{\rho}{\delta}\right)e^{-\rho(t-T)}-\frac{\rho}{\delta}e^{-(\delta+\rho)(t-T)}\right)^2$$

$$\dot{C}=C_T\left(-\rho\left(1+\frac{\rho}{\delta}\right)e^{-\rho(t-T)}+(\delta+\rho)\frac{\rho}{\delta}e^{-(\delta+\rho)(t-T)}\right)\left(\left(1+\frac{\rho}{\delta}\right)e^{-\rho(t-T)}-\frac{\rho}{\delta}e^{-(\delta+\rho)(t-T)}\right)$$

$$\dot{C}=\rho C_T\left(-\left(1+\frac{\rho}{\delta}\right)e^{-\rho(t-T)}+\left(1+\frac{\rho}{\delta}\right)e^{-(\delta+\rho)(t-T)}\right)\left(\left(1+\frac{\rho}{\delta}\right)e^{-\rho(t-T)}-\frac{\rho}{\delta}e^{-(\delta+\rho)(t-T)}\right)$$

$$\dot{C}=\rho\left(1+\frac{\rho}{\delta}\right)C_T\left(-e^{-\rho(t-T)}+e^{-(\delta+\rho)(t-T)}\right)\left(\left(1+\frac{\rho}{\delta}\right)e^{-\rho(t-T)}-\frac{\rho}{\delta}e^{-(\delta+\rho)(t-T)}\right)$$

$$\dot{C}=\rho\left(1+\frac{\rho}{\delta}\right)e^{-2\rho(t-T)}C_T\left(-1+e^{-\delta(t-T)}\right)\left(1+\frac{\rho}{\delta}\left(1-e^{-\delta(t-T)}\right)\right)$$

$$\dot{C}<0$$

since

$$1+\frac{\rho}{\delta}\left(1-e^{-\delta(t-T)}\right)>0$$

APPENDIX 13

We have

$$\dot{R} = R\left(\frac{1}{\sqrt{x}} - \frac{\dot{x}}{x}\right) - \frac{C}{x} \tag{86}$$

but now

$$\delta = -\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} \tag{87}$$

so

$$\dot{R} = -2\delta R - \frac{C}{x} < 0 \tag{88}$$

$$\dot{R} = -2\delta R - \frac{C_T\left(\frac{\left(\left(\sqrt{x_T}+\frac{1}{2\delta}\right)e^{\delta(t-T)}-\frac{1}{2\delta}\right)}{\sqrt{x_T}}e^{-(\delta+\rho)(t-T)}\right)^2}{\left(\left(\sqrt{x_T}+\frac{1}{2\delta}\right)e^{\delta(t-T)}-\frac{1}{2\delta}\right)^2} \tag{89}$$

$$\dot{R} = -2\delta R - \frac{C_T e^{-2(\delta+\rho)(t-T)}}{x_T} \tag{90}$$

with

$$x(t) \to x_T \equiv \frac{K_{ss}}{R_{ss}} = \left[\frac{1}{2\rho}\right]^2 \tag{91}$$

$$C(t) \to C_T = \omega\widehat{S}\left[\frac{\alpha}{\rho}\right] \tag{92}$$

$$\dot{R} = -2\delta R - \frac{\omega\widehat{S}e^{-2(\delta+\rho)(t-T)}}{\left[\frac{1}{2\rho}\right]} \tag{93}$$

$$R(t) = \omega\widehat{S}\exp\left(-2\delta t + 2\delta T - 2\rho t + 2\rho T\right) + e^{-2\delta t}D, \tag{94}$$

$$R(T) = \omega\widehat{S} + e^{-2\delta T}D = \omega\widehat{S}$$

$$D = 0$$

so

$$R(t) = \omega\widehat{S}e^{-2(\delta+\rho)(t-T)}$$

$$\frac{\dot{R}(t)}{R} = -2(\delta+\rho) < 0$$

APPENDIX 14

$$K = xR$$

so

$$\delta + \frac{1}{2}x^{-\frac{1}{2}} = +\frac{1}{2}\frac{\dot{x}}{x} \tag{95}$$

$$\frac{\dot{K}}{K} = \frac{\dot{x}}{x} + \frac{\dot{R}}{R} = \frac{\dot{x}}{x} - 2\left(\delta + \rho\right)$$

$$\frac{\dot{K}}{K} = \frac{1}{\sqrt{x}} - 2\rho$$

Since $\frac{\dot{x}}{x} > 0$ then $\frac{d}{dt}\left(\frac{1}{\sqrt{x}}\right) < 0$ so $\frac{1}{\sqrt{x(t)}} > \frac{1}{\sqrt{x(T)}} = 2\rho$ for all $t < T$ and therefore

$$\frac{\dot{K}}{K} > 0.$$

APPENDIX 15

Moreover substituting x and R yields

$$K\left(t\right) = x\left(t\right)R\left(t\right)$$

$$K\left(t\right) = \left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{\delta(t-T)} - \frac{1}{2\delta}\right)^2 \omega\widehat{S}e^{-2(\delta+\rho)(t-T)}$$

there exist a smooth path reaching K_{ss} at T is K_0 satisfies

$$K_0 = K\left(0\right) = \omega\widehat{S}\left(\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{-\delta T} - \frac{1}{2\delta}\right)e^{(\delta+\rho)T}\right)^2$$

Let $g\left(T\right) = \left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{-\delta T} - \frac{1}{2\delta}\right)e^{(\delta+\rho)T}$

$$g\left(T\right) = \left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{\rho T} - \frac{1}{2\delta}e^{(\delta+\rho)T}$$

$$g'\left(T\right) = \rho\left(\frac{1}{2\rho} + \frac{1}{2\delta}\right)e^{\rho T} - \left(\delta + \rho\right)\frac{1}{2\delta}e^{(\delta+\rho)T}$$

$$g'\left(T\right) = \frac{1}{2\delta}e^{\rho T}\left(\rho\left(\frac{\delta}{\rho} + 1\right) - \left(\delta + \rho\right)e^{\delta T}\right)$$

$$g'\left(T\right) = \frac{1}{2\delta}e^{\rho T}\left(\delta + \rho\right)\left(1 - e^{\delta T}\right) < 0$$

$$\frac{dK_0}{dT} = \omega\widehat{S}g\left(T\right)g'\left(T\right) < 0$$

References

[1] Beltratti, A., G. Chichilnisky, and G. Heal, 1998, Sustainable Use of Renewable Resources, in G. Chichilnisky (ed.), *Sustainability: Dynamics and Uncertainty*, Kluwer Academic Publishers, London, pp. 49-76.

[2] Clark, C. W., Clarke, F.H., and G. Munro, 1979, The Optimal Exploitation of Renewable Resource Stocks: Problems of Irreversible Investment, *Econometrica*, 47(1), pp. 25-48.

[3] Clarke, F.H., 1983, *Optimization and Non-smooth Analysis*, Wiley, NY.

[4] Clarke, F.H., and R.B. Winter, 1983, Local Optimality Conditions and Lipschitzian Solutions to the Hamilton-Jacobi Equations, *SIAM Journal of Control and Optimization*, 21,pp. 856-870.

[5] Dasgupta, P. S. and G. Heal, 1979, *Economic Theory and Exhaustible Resources*, Cambridge University Press, Cambridge.

[6] Docker, E., S. Jorgensen, N.V. Long, and G. Sorger, 2000, *Differential Games in Economics and Management Science*, Cambridge University Press, Cambridge.

[7] Long, N. V., and S. Katayama, 2002, Common Property Resource and Private Capital Accumulation, in G. Zaccour (ed.), *Optimal Control and Differential Games*, Kluwer Academic Publishers, London, 2002,pp. 193-209

[8] Pezzey, J., and C. A.Withagen, 1998, The Rise, Fall, and Substitutability of Capital-Resource Economies, *Scandinavian Journal of Economics*, 100(2), pp. 513-527.