2003RP-01

Real Options at Bell Canada

Marcel Boyer, Éric Gravel

Rapport de projet
Project report

Ce document a été produit dans le cadre du projet « Options réelles ».

Montréal
Février 2003

CIRANO
Centre interuniversitaire de recherche en analyse des organisations

CIRANO

Le CIRANO est un organisme sans but lucratif constitué en vertu de la Loi des compagnies du Québec. Le financement de son infrastructure et de ses activités de recherche provient des cotisations de ses organisations-membres, d'une subvention d'infrastructure du ministère de la Recherche, de la Science et de la Technologie, de même que des subventions et mandats obtenus par ses équipes de recherche.

CIRANO is a private non-profit organization incorporated under the Québec Companies Act. Its infrastructure and research activities are funded through fees paid by member organizations, an infrastructure grant from the Ministère de la Recherche, de la Science et de la Technologie, and grants and research mandates obtained by its research teams.

Les organisations-partenaires / The Partner Organizations

PARTENAIRE MAJEUR
. Ministère des Finances, de l'Économie et de la Recherche [MFER]

PARTENAIRES
. Alcan inc.
. Axa Canada
. Banque du Canada
. Banque Laurentienne du Canada
. Banque Nationale du Canada
. Banque Royale du Canada
. Bell Canada
. Bombardier
. Bourse de Montréal
. Développement des ressources humaines Canada [DRHC]
. Fédération des caisses Desjardins du Québec
. Gaz Métropolitain
. Hydro-Québec
. Industrie Canada
. Pratt & Whitney Canada Inc.
. Raymond Chabot Grant Thornton
. Ville de Montréal

. École Polytechnique de Montréal
. HEC Montréal
. Université Concordia
. Université de Montréal
. Université du Québec à Montréal
. Université Laval
. Université McGill

ASSOCIÉ AU :
. Institut de Finance Mathématique de Montréal (IFM^2)
. Laboratoires universitaires Bell Canada
. Réseau de calcul et de modélisation mathématique [RCM^2]
. Réseau de centres d'excellence MITACS (Les mathématiques des technologies de l'information et des systèmes complexes)

ISSN 1499-8610 (Version imprimée) / ISSN 1499-8629 (Version en ligne)

Real Options at Bell Canada

Marcel Boyer, *Éric Gravel*†

Résumé / *Abstract*

Nous débutons ce rapport en développant un exemple simplifié qui illustre l'importance de valoriser l'option de retarder un investissement. Une courte description des différentes options susceptibles d'être incorporées dans un projet d'investissement est ensuite donnée. Pour illustrer l'importance d'adopter un cadre d'analyse basé sur la méthodologie des options réelles pour la planification stratégique et l'analyse concurrentielle, nous présentons trois applications possibles d'options réelles dans l'évaluation d'investissements chez Bell Canada. La nécessité de l'adoption d'un tel cadre d'analyse dans le contexte de la réglementation des télécommunications fait ensuite l'objet d'une brève discussion. Nous terminons en soulignant que le succès de la mise en pratique d'un cadre «options réelles» dépend essentiellement d'un système efficace de collecte et de traitement de l'information. Deux appendices techniques fournissent plus de détails sur les techniques de modélisation et de solution qui sont couramment utilisées pour des problèmes d'options réelles.

Mots clés : Options réelles, valeur d'option, volatilité, risque, irréversibilité, télécommunications.

In this report, we first develop a simplified example that illustrates the importance of considering the option ``waiting to invest'' when valuing an investment. This is followed by a short description of other options that could be embedded in an investment opportunity. In order to stress the importance of the real option mind-set in strategic planning and competitive assessment, we present three examples of possible applications of real options for evaluating investments at Bell Canada. A brief discussion follows on the importance of a real options mind-set in the telecommunications regulation context. Finally we conclude by underlining the importance of an efficient information gathering and processing framework to implement a real options framework. Two technical appendices provide more details on both the modeling and the solving techniques that are commonly used to implement real options.

Keywords*: Real options, option value, volatility, risk, irreversibility, telecommunications.*

---

* Marcel Boyer, titulaire, chaire Bell Canada en économie industrielle, Université de Montréal, Fellow, CIRANO.

† Professionnel de recherche, CIRANO.

Contents

1 Introduction

An investment opportunity typically includes a certain number of options that can be exercised over time as new information is continuously gathered and as exogenous uncertainty evolves or is resolved. Furthermore, resources that are committed to a project often cannot be recuperated, that is, an investment is usually or at least in good part irreversible. In such a case, a decision criteria that minimizes the probability of landing in an unfavorable state is warranted. This is the essence of the real options approach to capital budgeting.

The real options method extends the standard net present value method (NPV) by recognizing that high level managers have the flexibility to intervene at certain points in the future as new information becomes available and as uncertainty evolves. It is also an uncertainty and risk management tool that efficiently uses available information to diminish (without necessarily eliminating) the risk of loosing valuable resources.

Among the different options that are available to a decision maker, we find: the options of delaying an investment, the operating flexibility options, the time to build options, the options to switch and the options to abandon an ongoing investment. In real life situations, several of those real options can be embedded in a given project. With the arrival of new information, a decision maker has in many cases the flexibility to modify in some way an investment project.

Rather than considering an average scenario or a decision tree that examines only a subset of flexibility points as in “advanced” net present value

(NPV) analysis, a real options approach explicitly values flexibility. Hence, it adds to the fundamental financial and economic value of the project the real value of managerial discretionary interventions. Rules of thumb such as augmenting the discount rate for a project that is perceived to be riskier or to generate payoffs that are far into the future also fail to fulfill the requirements set out and met by a rigorous real options formulation.

The effect of managerial flexibility on value is non-linear, a feature that NPV, advanced or not, fails to recognize in a proper way. If at some point in time the project fundamentals move on a bad or unfavorable path, a high level project manager may be able to intervene and decide to abandon the project. Hence, the future negative value of the project if and when such a point is reached will never materialize. However, the time at which such a point may be reached is itself random. Hence, simple standard discounting methods cannot be applied.

In the real options approach, a project is valued as a set of optimally exercised options over time. The evaluation is truly dynamic. At some points or times in the future, the decision maker can and will intervene. For instance, he may be able to delay or advance the realization of and investment project. In doing so, he must weigh the benefits of waiting for new information against the foregone profits due to investment delay. Evaluating the project is evaluating the optimal *strategy* of intervention, that is, the optimal *decision rule* that yields the highest expected net present value. Rather than evaluating a project by projecting uncertain cash flows and discounting their expected value to obtain the net present value, the real options approach considers each investment in the different phases of a project as the striking or exercise price of an option to proceed to the next phase. The impact on investment evaluation could be very significant.

The impact of valuing an investment by a real options approach is that some projects which were considered to be unprofitable (negative NPV) may turn out to be profitable when flexibility is explicitly valued. Consequently, real options rigorously forces the organization not to discard projects that have future value creation potential, but that are currently unprofitable according to standard criteria. Similarly, in the context of mutually exclusive investments, a less profitable opportunity may be chosen if the valuation method fails to recognize flexibility, thereby destroying potential value for the firm.

It is therefore important for high level executives to understand the main methodological elements and steps of sound real options analysis in order to embrace its potential as a mind-set for strategic decision making.

In the next section, we develop a simplified example that illustrates the importance of considering the option "waiting to invest" when valuing an investment. This is followed by a short description of other options that could be embedded in an investment opportunity. The following section stresses the use of the real option mind-set in strategic planning and competitive assessment. We then present three examples of possible applications of real options for evaluating investments at Bell Canada. A brief discussion follows on the importance of a real options mind-set in the telecommunications regulation context. Finally we conclude by underlining the importance of an efficient information gathering and processing framework to implement a real options framework. Two technical appendices provide more details on the modelling and solving techniques that are commonly used to implement real options.

2 Real Options and the Shortfalls of the Traditional Capital Budgeting Methods

2.1 Shortfalls of the Standard Net Present Value

Traditionally, managers have used the standard net present value (NPV) method to make capital budgeting decisions. The analysis below can also be applied to the internal rate of return (IRR) method or to other "advanced" NPV or IRR methods. In this section, we will argue that for several types of investment opportunities the NPV method is incomplete because it fails to properly value managerial flexibility as well as the other embedded investment opportunities in a world where uncertainty and information evolve over time. Different assumptions, implicit in the standard NPV method, render the method incompatible with several real life situations.

To be operational, the standard NPV requires three inputs, these are:

1. the present value of the project costs I,
2. the project's sequence of expected cash flows $\{C_t\}_{t=n}^{N}$, and
3. the risk adjusted discount rate ρ.

To justify an investment, the standard NPV requires that the difference between the expected discounted value of the project's cash flows (DCF) and the expected discounted value of the cost (I) of the project be superior or equal to zero. The standard NPV criterion can then be written as

$$\text{decision} = \begin{cases} \text{Invest if} & DCF - I \geq 0 \\ \text{Do not invest if} & DCF - I < 0 \end{cases}. \tag{1}$$

Under the spell of uncertainty, the above decision rule is adequate only under specific circumstances. We start by considering a simple project that cannot be altered once it is operational and where no other profit opportunities arise from realizing the original investment. In this case, (1) is a satisfactory criterion if one or both of the following criteria is satisfied:

1. the manager can later reverse his decision and recuperate I at a negligible cost;
2. the investment is a now or never opportunity.

If I can be recuperated when market conditions turn out to be unfavorable, we say that the investment is perfectly reversible. It is not difficult to imagine situations in which this first condition is violated. For example, consider an investment in an optical fiber network. It is hard to imagine that the network can be closed and dismantled without loosing a substantial part of the original investment. If the manager has the option to delay, we must not ignore the possibility of realizing the project at a subsequent date. At this point, a simple example of an irreversible investment that violates the now or never condition will illustrate the above principles.

Consider a firm that has the opportunity to realize an irreversible investment today (year $t = 0$) or in the future (at year $t \geq 1$) at a constant cost of $1,600M\$$. Suppose that the investment allows the firm to produce a million units of a certain good forever beginning in the year after the investment is undertaken, and that the hurdle rate of return required from that investment is 10% (discount rate). Furthermore, to simplify the example, suppose that there are no production costs and that the price of a unit evolves according to the following diagram:[1]

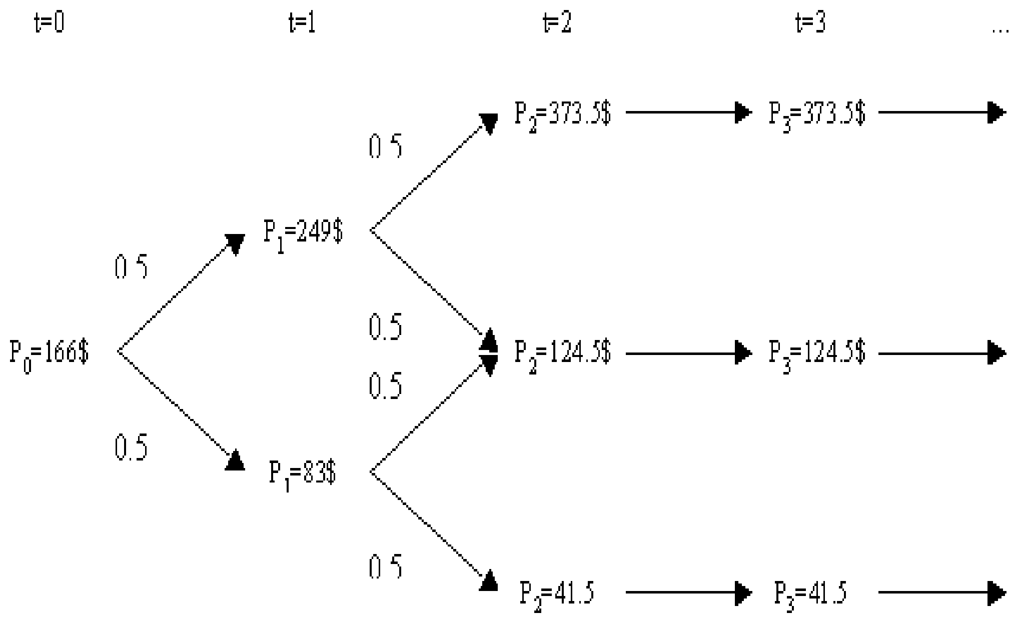

Figure 1. Evolution of prices

According to figure 1, we have the following information at time 0:

1. the uncertainty surrounding the price of the firm's output is totally resolved in year 3 ($t = 3$);
2. the price of the output today (year 0) is known to be equal to 166$;
3. the price of the output during year 1 and year 2 will increase will increase by 50% with probability 1/2 and decrease by 50% with probability 1/2.

[1] The price at each node is also equal to the expected value of the future prices.

For example, if the price of a unit is 249\$ in year 1, there is a probability of 1/2 that it will rise to 373.50\$ and a probability of 1/2 that it will fall to 124.50\$ in year 2.[2] The same pattern is observed at the other nodes until the market stabilizes in year 3 at one of three possible price levels.[3]

In periods 0, 1 and 2, the firm can either invest or postpone the decision to invest or not till the following year. If the manager ignores the fact that the investment is irreversible and that it is possible to wait till the following year, he will invest immediately because (Note that at time 0, the expected cash flow is 166\$ in all periods.)

$$NPV = -1600M\$ + \sum_{t=1}^{\infty} \frac{166M\$}{(1.1)^t} = 60M\$ > 0 \;\Rightarrow\; \text{invest.}$$

If the manager considers the option to wait before investing, he will use the decision criterion:

$$\text{decision at } t = \max\{\text{invest at } t, \text{ wait till } t+1\} \tag{2}$$

Choosing the period to invest is equivalent to comparing a series of mutually exclusive projects where each project represents the investment realized at a different period. But it is imperative to consider in doing so that information is changing as time goes by.

We will now compare the value at $t = 0$ of the dynamic and static strategies, the latter being given by . To solve this problem, the technique of

[2]Consequently, the expected value of the price in $t = 2$, conditional on the fact that it has reached 249 in year 1, is also equal to 249\$ $(0.5 \cdot 373.50\$ + 0.5 \cdot 124.50\$ = 249\$)$.

[3]At time 0, the probability that the price will stabilize at 373.50\$ is 1/4, at 124.50\$ is 1/2, and at 41.50\$ is 1/4. If in year 1, the price happens to be 83\$, then the probable prices in year 3 will be revised, conditional on the fact that the price is 83\$ in year 1; the revised probability that the price will stabilize at 373.50\$ is 0, at 124.50\$ is 1/2, and at 41.50\$ is 1/2.

dynamic optimization will be used and we shall proceed recursively starting at $t = 2$. The following notation will be used:

$$\begin{aligned} NPV_t &\equiv \text{expected net present value at } t \text{ if the firm invests at } t, \\ B_t &\equiv \text{expected present value at } t \text{ if the firm postpones the decision} \\ &\quad\ \text{to invest or not to the next period.} \end{aligned}$$

According to figure 1, the market is stable after $t = 2$ and the project then becomes a now or never proposition. Postponing the investment in this case has no other effect than diminishing the present value of the project and the decision will be

$$\begin{aligned} \text{if } P_2 &= 373.50\$, \text{ then} \\ & \qquad NPV_2 = -1,600M\$ + \sum_{i=3}^{\infty} \frac{373.50M\$}{(1.1)^{i-2}} = 2,135M\$ > 0 \Rightarrow \text{ invest} \\ \text{if } P_2 &= 124.50\$ \text{ or } 41.50\$, \text{ then } NPV_2 < 0 \Rightarrow \text{ do not invest.} \end{aligned}$$

At $t = 1$, the firm can either invest immediately or wait until $t = 2$. The problem is easily solved at $P_1 = 83\$$ where the NPV of the investment is negative in all future contingencies. The investment is worthless in that situation. At $P_1 = 249\$$ we must compare both strategies. By investing immediately the expected NPV is

$$NPV_1 = -1,600M\$ + \sum_{i=2}^{\infty} \frac{249M\$}{(1.1)^{i-1}} = 890M\$.$$

If the manager waits, the expected present value of this strategy is

$$\begin{aligned} B_1 &= \left[0.5\left(\max\left\{0,\ -1,600M\$ + \sum_{i=3}^{\infty} \frac{373.50M\$}{(1.1)^{i-2}}\right\}\right)\right. \\ &\qquad \left. +0.5\left(\max\left\{0,\ -1,600M\$ + \sum_{i=3}^{\infty} \frac{124.50M\$}{(1.1)^{i-2}}\right\}\right)\right] \cdot \frac{1}{(1.1)} \\ &= 970.45M\$. \end{aligned}$$

Because $B_1 - NPV_1 > 0$ it is optimal to postpone the investment and wait for a favorable realization at $t = 2$. By waiting the firm can avoid realizing at $t = 1$ an investment which will have a negative value, ex post, if P_2 turns out to fall to 124.50\$ at $t = 2$.

Let us now consider the optimal decision at $t = 0$. By investing immediately the expected NPV is

$$NPV_0 = -1,600M\$ + \sum_{i=1}^{\infty} \frac{166M\$}{(1.1)^i} = 60M\$.$$

If the manager waits, the value of waiting is

$$\begin{aligned} B_0 &= [0.5 \max \{NPV_1 \text{ if } P_1 = 249\$,\ B_1 \text{ if } P_1 = 249\$\} \\ &\qquad +0.5 \max \{NPV_1 \text{ if } P_1 = 83\$,\ B_1 \text{ if } P_1 = 83\$\}] \cdot \frac{1}{(1.1)} \\ &= \frac{0.5 \cdot 970.45M\$ + 0.5 \cdot 0M\$}{1.1} = 441.11M\$. \end{aligned}$$

At $t = 0$, because $B_0 - NPV_0 > 0$, it is thus optimal to wait till next period before a decision is made. If at $t = 1$, the price falls to 83\$, the investment project should be abandoned. If on the other hand the price increases to 249\$, then it will be optimal to wait till $t = 2$ because in such a state (price level), postponing the investment by one more period, up to $t = 2$, gives the largest net present value.

The difference $B_0 - NPV_0$ is the value of the option to wait. The passage of time increases the information available to the decision maker. The possibility of avoiding an unfavorable situation affects the value of an investment. Consequently, investing immediately has an opportunity cost that is equal to the value of the extra information provided by the passage of time. Delaying an investment also reduces the present value of the necessary cash outlay.

The cost associated to waiting is the revenues lost during the waiting period. The goal of the dynamic approach to investment is to balance these benefits and costs to obtain an optimal investment strategy conditional on the information available. This optimal strategy gives the best course of action (invest or wait) in all future states.

Equivalently, by adding the value of the option to invest at a later date to the direct cost of the investment, we can define an "improved NPV" that considers the opportunity cost of investing immediately. Here NPV_0 can be decomposed into

$$NPV_0 = DCF_0 - I, \tag{3}$$

it is optimal to invest right away if we have

$$NPV_0 - B_0 > 0. \tag{4}$$

By combining (3) and (4) we get the optimal rule: invest now if and only if

$$DCF_0 - I - B_0 > 0. \tag{5}$$

According to (5), for investing today to be the optimal decision, the discounted expected cash flows from investing today must exceed the sum of the cost of the investment (I) and the value of the optimally managed option to invest at a later date. By making the investment today, one renounces to the option of making the investment at a later date once more information has been gathered. This renouncement has a cost, namely the value of the option that is exercised.

The above example is fairly simple, but it is sufficient to show that the standard NPV criteria may lead to decisions that are suboptimal when there is uncertainty in the variables governing the value of a project, new information will become available, and the investment is in good part irreversible.

Value may be lost if the decision criterion does not properly take into account the dynamic structure of the problem. Options are usually embedded in an investment opportunity and valuation procedures must be able to take into account the value that stems from active management in exercising those option. The expansion of the information set through time may create an incentive to wait for new information before investing in order to reduce the exposure to downside risk.

Other capital budgeting methods attempt to deal with uncertainty and flexibility in the decision making process. Although these methods are a step in the right direction, they are often incomplete. We have in mind here sensitivity analysis and decision-tree analysis. Sensitivity analysis attempts to asses the impact of uncertainty on the NPV of an investment but it does not consider flexibility in the decision making process. For its part, decision-tree analysis can include flexibility but it quickly becomes intractable when the number of possible states of the project's economic environment increases along with the flexibility points.

The possibility of waiting to invest is one of the many options available to the decision maker that are not accounted for with the standard NPV criterion. The following section is a description of the most common real options typically present in investment projects.

2.2 Common Real Options

As mentioned in the previous section, flexibility has a value that is not captured by standard NPV analysis. The possibility of postponing a decision and wait for new information can significantly alter the value of a firm if an unfavorable situation can be avoided. With an example we showed that the standard NPV rule must be adjusted to account for the fact that by investing now we renounce to (exercise) the option of waiting for new information.

Waiting to invest is not the only way to manage uncertainty and profit from extra information. Flexibility is sometimes found in subsequent steps of the decision process. This section describes other situations in which real options analysis can be useful.

2.2.1 Operating Flexibility Options

The presence of operating costs confers an option when production can be temporarily suspended at a negligible cost. A value maximizing manager will only produce when the price P is superior to operating (variable) costs C. The present value of each instant's production decision is then equivalent to a real call option with a payoff equal to

$$\max\left[P - C, 0\right].$$

At each instant, the firm has the option to expense C (strike price) in exchange of P (value of the underlying). The expected present value of a project's cash flows is equal to the sum of each instant's operating options. In this case, both P and C may be stochastic. The standard NPV criteria implicitly assumes that production goes on uninterrupted.

2.2.2 Time to Build Options

Certain projects necessitate a series of cash outlays before they generate any cash flows. Building usually takes time and it is possible to delay or abandon the project before it is completed. This class of investment opportunities includes research and development (R&D) ventures and capital intensive projects that take a substantial amount of time to complete (optical fiber network expansions, construction of power lines, etc.).

Realizing one of the project's stages is like exercising an option that has a payoff equal to the value of the option to undertake the next step. The value of each of these options ultimately depends on the value of the investment once completed. The valuation procedure must take into account the fact that the manager has the option to delay or abandon the project if market conditions turn out to be unfavorable. The standard NPV method does not account for this flexibility. It is as if the project has to be completed without interruption.

In this situation, information can manifest itself in the price of the output, in the production costs and/or in the project's remaining cost of completion. The possibility of a significant event that can jeopardize or increase the viability or value of the project can also be included.

2.2.3 Compound Options

Investment opportunities that provide immediate cash flows and future investment possibilities that would be otherwise unrealizable can be included in this class. An initial project can serve as a stepping stone towards entering and developing (a decision to be made later) profitable new markets. If the evaluation method ignores or does not value properly the embedded options, valuable investments can be rejected by the standard NPV method that considers a project on a stand-alone basis.

2.2.4 Options to Switch

A technology that allows the substitution between different inputs or the substitution between several outputs to be produced is a bouquet of switching options. A direct example is a boiler that can alternate between different types of fuels to produce electricity. At each moment, the decision maker may be expected to exploit the most profitable alternative. A flexible technology is undervalued if it is compared to a dedicated technology without properly valuing the options related to production flexibility. The standard NPV method is too rigid in this case. The asset must properly be valued as a portfolio of switching options.

2.2.5 Options to Abandon

With operating flexibility options, operations can be stopped and restarted at a negligible cost, thereby allowing a reduction or elimination of operational losses. If it is costly or impossible to restart operations, one has to consider the possibility of abandoning current operations in exchange for a salvage value. With stochastic revenues and costs, a firm may be willing to incur losses before abandoning a project in order to avoid being out of the market in the event of an upturn. The degree of the loss tolerance is a function of the possibilities of re-entering the market after an exit.

The value of such an investment is equal to the expected present value of its cash flows plus an abandonment option. The standard NPV method implicitly assumes that operations go on uninterrupted during the productive life of the asset.

2.2.6 Real Options Combinations

In real life situations, several of the aforementioned real options can be embedded in a specific project. One must consider the impact of all the options embedded in the decision making process.

The Boiler example[4]

As a first illustration, we shall describe the case of a firm that considers the opportunity of acquiring a boiler to satisfy its energy needs. Currently, these energy needs are fulfilled by purchasing electricity from outside sources. The firm has the option to switch from outside electricity to internal boiler produced energy. Furthermore, we suppose that purchasing a boiler is an irreversible investment and that the prices of oil P_{oil}, gas P_{gas} and electricity $P_{electricity}$ are all stochastic.

The firm has access to three different boiler technologies. The first boiler burns natural gas and costs $63,500$\$ to build. Its efficiency-adjusted price of fuel, defined as the spot price of the relevant fuel times a factor that reflects the thermal efficiency of the boiler, is $1469P_{gaz}$. The second unit burns oil and costs $66,600$\$ to build. Its efficiency-adjusted price of fuel is $1408P_{oil}$. Finally, a third boiler that is capable of burning oil or gas is available and costs $68,700$\$ to build. It costs S to switch between oil and gas and the efficiency-adjusted price are the same as for the dedicated boilers, that is $1469P_{gaz}$ and $1408P_{oil}$ for gas and oil respectively. Hence, when the ratio of the price of oil over the price of gas is superior to 1.04 (1469/1408), it is less costly to operate with gas.

[4] See Amran and Kulatilaka (1998), chapter 16.

Two options can be identified in this problem. First, the firm can switch from outside electricity to boiler produced energy by incurring the sunk cost related to building any of the three boilers. This is a waiting to invest option and the optimal decision rule will be based on relative fuel and electricity prices. The second option arises from the fact that the third technology permits switching between fuels at a fixed cost of S. When prices are stochastic, this extra flexibility is a hedge that is not available with a single fuel technology. The value of this flexibility may more than offset the higher investment cost of the dual fuel boiler. If the valuation method does not consider this extra flexibility, a suboptimal technology may be chosen.

In Figure 2 we illustrate the value of this flexibility. The value P_g is the ratio P_{oil}/P_{gas} above which it is optimal to change from oil to gas while the value P_o is the ratio P_{oil}/P_{gas} below which it is optimal to change from gas to oil, when the flexible technology is in place. If gas is currently used, it is not profitable to switch to oil as long as the ratio P_{oil}/P_{gas} is superior to P_o. Similarly, if oil is currently used, it is not profitable to switch to gas as long as the ratio P_{oil}/P_{gas} remains below P_g. Between these two critical values, the difference between the observed ratio P_{oil}/P_{gas} and the indifference ratio 1.04 is not large enough to offset the fixed costs S related to switching fuels.

When the ratio is superior to the value PP_g, the price of oil relative to gas is so high that the lower cost of the dedicated gas boiler dominates the total cost (investment cost + the cost of forfeiting the option to switch between fuels) of the flexible technology. This justifies the use of the gas boiler. When the ratio is inferior to the value PP_o, the oil boiler is more advantageous. Between PP_o and PP_g, it is hard to know if the future ratio will be above or below 1.04; in this situation flexibility is very valuable, justifying the use of the dual fuel technology in spite of its higher investment cost.

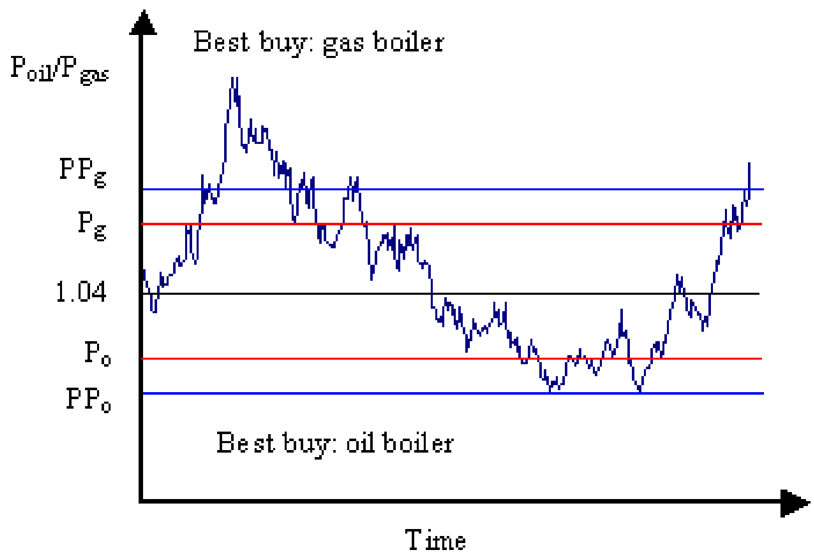

Figure 2. Sample of the Price Ratio and Switching Rules

The above analysis is the first step. The optimal technology as a function of P_{oil}/P_{gas} is now known. As we mentioned above, the firm is currently buying electricity from outside sources. Consequently, the second part of the problem is to determine the optimal timing of a boiler investment. The optimal timing rule is a function of the price of electricity relative to the price of oil and gas.

As illustrated in Figure 3, it is not advantageous to build a boiler if the price of electricity is low. When the price of electricity rises above some critical value, it is preferable to operate a boiler. Each of the three curves in Figure 3 characterize for each boiler the relative prices of electricity, oil and gas that justify investment. Consequently, before any boiler is built, the optimal decision rule is a function of the variables P_{oil}/P_{gas} and $P_{electricity}/P_{oil}$ that yields a choice from the set

$$\{\text{wait, build an oil boiler, build a gas boiler, build a flexible boiler}\}.$$

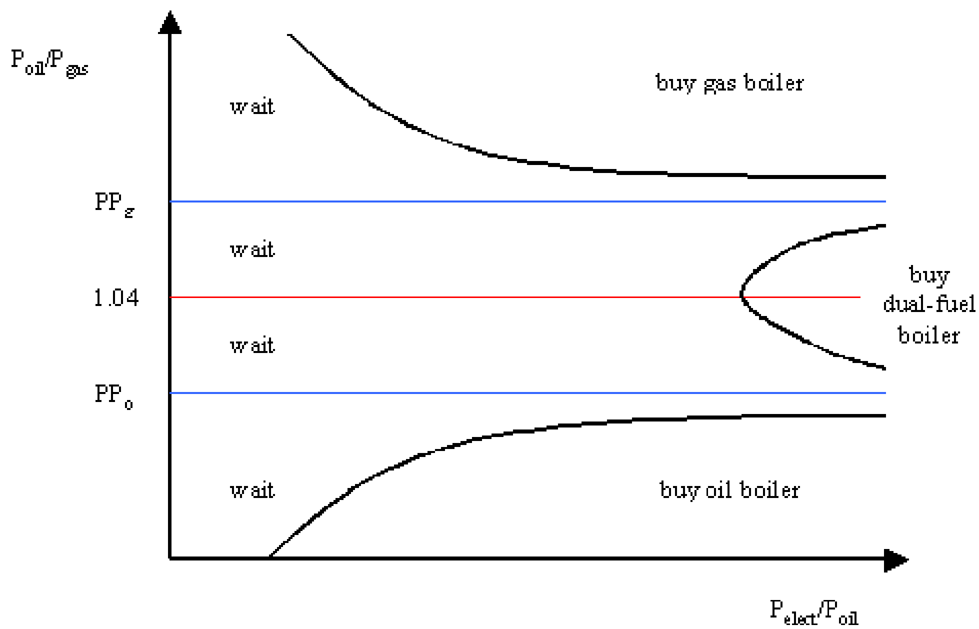

Figure 3. Boiler Buying Rules

With the real options methodology, the objective is to diminish the probability of being in a situation where electricity prices fall to a level that do not justify the purchase of a boiler. In this example, the decision maker must use all the available information to avoid losing the sunk costs of a boiler and to choose the most suitable technology. It is important to identify and value all the options that are relevant to a particular investment opportunity in order to make the optimal decision.

The following example illustrates how the standard NPV method can induce the holder of an investment opportunity in error.

The Portlandia example[5]

Portlandia Ale is a want-to-be start-up microbrewery with no products on the market yet. Portlandia must invest 0.5 million dollars per quarter for the next two years and another 12 million dollars at the end of these two years to develop and launch its first product in order to gain an established microbrewery status. Currently, management evaluates that the established company will be worth 22 million dollars in two years, needless to mention that this value is highly uncertain. We suppose here that the risk free interest rate and the risk-adjusted discount rate are equal to 5% and 21%, respectively. Furthermore, for this example, continuous compounding will be used.[6]

If Portlandia managers use the standard NPV method, they find that the NPV is negative. That is

$$NPV = 22e^{-0.21 \cdot 2} - \left\{ \left(0,5 \cdot \sum_{t=0}^{7} e^{-0.05 \cdot 0.25 \cdot t} \right) + 12e^{-0.05 \cdot 2} \right\} = -0,23M\$ < 0,$$

meaning that the project is rejected.

It is fairly easy to argue that the above valuation method is inadequate. It is supposed in the above calculation that no matter how market conditions evolve in the next two years, Portlandia is committed to spending the 12 million dollars necessary to lunch the product. In reality, if the future value of the company falls below 12 million dollars at the end of year 2, the investment to launch it should not be made. The launching sunk investment cost can be avoided if market conditions are unfavorable. By spending the 0.5 million dollars per quarter, Portlandia acquires the option to spend the extra 12

[5] See Amran and Kulatilaka (1998), chapter 10.
[6] See appendix A for details.

million dollars to obtain the future value of the firm.

If V_2 denotes the value of the firm in two years, the payoff of the option to invest is equal to

$$\max\left[V_2 - 12, 0\right].$$

This situation is equivalent to a European Option valuation problem. If we use the Black-Scholes framework with a value volatility parameter of 40%, the value of the option to invest in two years is equal to[7]

$$22e^{-0.21\cdot 2}N\left(d_1\right) - 12e^{-0.05\cdot 2}N\left(d_2\right) = 4,96$$

with

$$d_1 = \frac{\log\left(\frac{22e^{-0.21\cdot 2}}{12}\right) + \left(0.05 + \frac{1}{2}\cdot 0.40^2\right)\cdot 2}{0.40\sqrt{2}}$$

and

$$d_2 = \frac{\log\left(\frac{22e^{-0.21\cdot 2}}{12}\right) + \left(0.05 - \frac{1}{2}\cdot 0.40^2\right)\cdot 2}{0.40\sqrt{2}}.$$

Consequently, the value of the option to realize the launch step is equal to 4,96 million dollars and to acquire this option, one must invest 0.5 million dollars per quarter for two years, that is 3,83 million dollars in present value. The value of the properly defined project at $t = 0$ is therefore positive at 1,13 million dollars.

In this case, if the standard NPV criterion is used to decide to invest or not, Portlandia loses a valuable investment opportunity. The real options approach considers the fact that management can avoid future sunk costs in unfavorable situations. It values the opportunity to invest by considering the

[7] Where $N\left(\cdot\right)$ is the cumulative distribution function for a standardized normal random variable.

future evolution of the company value. Because the set of possible values is continuous, a decision-tree analysis in this case is very cumbersome.[8]

Finally, to be even more realistic, the project can be valued as a set of compound options. Each 0,5 million dollar outlay is equivalent to purchasing the option to expend the next amount. In this case, part of the intermediate expenses can be avoided along with the 12 million dollar outlay. This extra flexibility gives even more value to the initial project.

[8]In general, not only is a decision tree approach cumbersome because of the continuity of the state space, but in most cases, the times at which the different decision nodes are reached are themselves stochastic variables. This makes the decision tree approach almost inapplicable.

3 Real Options, Strategic Planning and Competitive Interactions

The real options mind-set is particularly well suited for the analysis of strategic decisions regarding significant investments, acquisitions, mergers and alliances, technology development and R&D programs, re-engineering and restructuring, etc. There are real options embedded in those decisions: they should not only be recognized and evaluated in an appropriate and rigorous way but they should also be developed and built into all major strategic project. The value of the firm depends crucially on the management of real options.

Strategic planning is first an exercise in managing flexibility, that is in literally building real options into the future of the firm, and second in characterizing the optimal decision rules to profitably exercise those real options. Building real options and characterizing the their optimal exercise rules can be materialized as specifying future decision nodes, whose time of emergence is typically stochastic, at which some steps (producing, shutting down, delaying, expanding, contracting, abandoning, switching, etc.) may or may not be taken. Drawing a strategic plan is an active exercise in anticipating but also shaping the future of the firm's environment. It is an exercise in making sure that the decision maker will be able to fully benefit from the stochastic situations to emerge, whatever those situations may turn out to be. To be bale to achieve such a position, managers must create flexibility and optimally manage it. This pro active role of high level managers in determining the future of the firm is in some sense their most important task.

The value of strategic planning itself is determined by the quality, significance and value, of the real options designed and imbedded in the plan and by the quality of the evaluation procedure of those real options, including the management information system underlying the evaluation. It is in this precise sense that the design and management of real options, through the exploitation of volatility in the firm's environment and irreversibility in many strategic decisions, create value for the firm.

For some observers, the role and value of higher level executives in a corporation should be defined and understood around the building and evaluation of the real options entering into the strategic planning of the firm's future. Building, identifying and evaluating real options may represent in this sense the most important responsibilities of the higher level executives.[9]

Among investment decision tools, real options theory is rapidly gaining reputation and influence. Although specialists warn against its often daunting complexity, they also stress its unique ability to take account of future flexibility and the importance of future moves and decisions in valuing current investments.

The real options approach emphasizes the indivisibility and irreversibility of investments. Indivisibilities often imply a limited number of players, hence imperfect competition. Yet, while it is often stressed that real option theory is best to analyze investments of strategic importance, and the word "strategic" appears repeatedly in the real-options literature, the bulk of that literature involves decision makers playing against nature rather than against other rational players, that is, facing a non-reactive business environment rather than an environment characterized by the presence of aggressive competitors. The analysis of strategic considerations, in a game theoretic sense, is still

[9] See Christoffersen and Pavlov (2003).

in its infancy and should be high in the real-option research agenda. The proper way to jointly manage competitive interactions and real options is a particularly demanding challenge that high level managers must be prepared to face.

Strategic (oligopolistic) competition can force a decision different from the one prescribed by a pure real options analysis. There may be a first mover advantage when two or more firms hold the same investment opportunity. Preemption motives must be introduced in the analysis. It may be optimal to act faster than what might be prescribed by a real options analysis. For example, if two firms contemplate entering into a risky stochastic natural monopoly industry where the first mover gains the entire (but unknown) market, it is clear that real options cannot be blindly applied without considering the rival's potential actions. In the above case, the value of the investment depends on the competitor's actions or strategy.

The same reasoning applies to a market where network effects are significant, that is, a market where the value of products and services for the consumer increases with the number of other users. A preemptive strike might ensure a client base large enough to deter a competitor's entry. In such cases, the incentive to enter the market first can be in conflict with what might be prescribed by a real options analysis. This does not mean that real options should be ignored, but rather that the model must be formulated in a game theoretic context.

The presence of a second mover advantage must also be included in the problem's formulation. Letting the competitor act first may in some cases increase the value of the follower's investment opportunity. For example, if demand is very uncertain in a particular market, letting the competitor act first can yield information about demand that reduces the follower's risk of committing resources to an unprofitable venture. The decision maker must be aware of which context is the appropriate one when applying a real options approach.

Real options, contrary to financial options, may have a negative value. As mentioned before, the value of real options derives from the active management of project flexibility as new information is acquired and exogenous uncertainty unfolds over time. However, the possibilities of modifying the planned course of a given project imply that the firm's commitment to develop and eventually complete the project is relatively low. This lack of commitment may invite more aggressive behavior from competitors whose objective may be to drive the firm out of the project or market, or more aggressive attacks from the opponents to the project. Active management means that some options should be closed (or exercised) while others should be kept open. It is a major responsibility of high level executives to identify which options should be closed in favor of a strong commitment to complete a project and which options should be kept open in order to be more flexible in order to benefit from more and better information as well as reduced uncertainty as time goes by.

It is important to keep in mind that real options analysis must be applied with caution. In the preceding sections we stressed the importance of adapting the investment decision process to the specific features of the project. With equal importance, the firm's competitive setting must be accounted for. Otherwise, we may end up employing a tool that is no better than the standard NPV in the non-competitive context.

4 Potential Real Options at Bell Canada

The telecommunications industry presents several opportunities for employing the real options methodology. In this section, we present three cases: an optical fiber network expansion project, a research and development project, and an information technology investment. An appendix presents in more details the methods used to model these investment decisions.

4.1 Optical Fiber Network Expansion

Real options can help determine the optimal timing of an optical fiber network expansion along with the value of the opportunity to invest. We can view an expansion as a sequential investment with the following steps:[10]

1. acquire the rights to lay optical fiber in the ground,
2. install ducts,
3. acquire optical fibers and pull them through the ducts, and
4. acquire and install the systems needed to transfer data through the fibers.

Each step can be viewed as an option with a payoff equal to the value of the option to undertake the next step. Because of this, the opportunity can be valued as a set of compound options. For this model, we suppose that each step can be delayed and that the firm can abandon midway through

[10] See Lassila (2001).

the expansion if market conditions turn out to be unfavorable. If each phase takes time to complete, building delays can be introduced into the model. With these building delays, an option to abandon during construction can be included into the analysis. The complexity of the model increases with the number of flexibility points, more flexibility entails more embedded options.

The simplest way to treat this kind of problems is to suppose that:

1. only the value of an operational network is stochastic, and
2. each step can be realized instantly (no building delays).

The above hypotheses lead to a decision rule that is function of the value of the underlying operational network. For each of the four phases of the expansion, a threshold value marks the boundary between investing and waiting. When the stochastic state variable (value of the operational network) crosses the threshold, it is optimal to invest. In this case, the analysis shows that the first threshold value is superior to the second, the second is superior to the third and the third is superior to the fourth. Consequently, each step of the expansion will be completed as soon as the value of the operational network crosses the first stage threshold.

An equivalent to the above formulation would be to merge all four stages into one and value a single waiting to invest option. If the network can be built instantly, there is no interest in postponing the later stages because it would go against the value maximization rule. Consequently, we simply have a single waiting to invest option valuation problem.

It is difficult to imagine that the realization of an optical fiber network expansion is instantaneous, because of this, the results of the above model are of limited interest. The analysis is considerably enriched if we include building delays by imposing a physical constraint that limits the speed at which each phase can be realized. In this framework, we can also suppose that the total cost of the expansion is now a stochastic variable since it is very likely that costs are also subject to unpredictable fluctuations.

For the model to be tractable, we impose the following:

1. each investment outlay is sunk,
2. previously installed capital does not decay,
3. and a unit is not productive until the project is completed.

As in the previous formulation, realizing part of the project is equivalent to exercising the option to undertake the next step. The above decision process is illustrated on Figure 4 where V represents the present value of the fully operational network, I_m the maximum amount of investment allowed at each period (imposed by the physical constraint), T the uncertain duration of the expansion and t the time at which it is optimal to start investing.

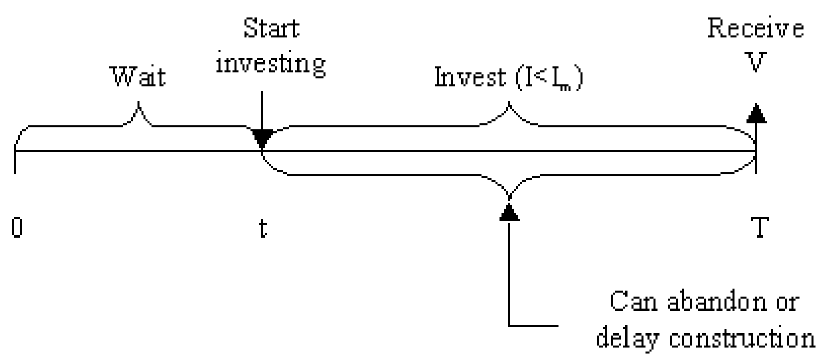

Figure 4. Decision process for an optical fiber network expansion.

For this model, two state variables influence the value of the investment opportunity, these are:

1. the amount of investment required to complete the project, and
2. the stochastic output price.

The analysis yields an optimal investment rule that is a function of both the output price and the random amount of investment required to complete the project. For each amount of investment required to complete, a threshold output price corresponds to the minimum price that justifies investment. Furthermore, at the threshold, the marginal benefit of investing is equal to the marginal cost.

With this information, the manager can decide if and when it is optimal to start building the network and once a step is completed, if and when it is optimal to proceed with the next one. The relation between the optimal investment threshold and volatility is as before. Price volatility increases the incentive to wait in order to avoid investing in what may turn out to be ex post a more probable unfavorable situation.

In this case, the method values the fact that management can stop investing if market conditions worsen due to larger costs or a depreciated project value. The standard NPV is usually computed, at least implicitly, by supposing that construction will go on uninterrupted even if the context eventually does not justify the investment. The suspension option captures the value of being able to avoid future sunk costs in an unfavorable situation. Finally, if the actual context does not justify the investment, the project is not necessarily worthless. There is a probability that future market conditions will improve and that the option of being able to realize the expansion at that moment has a value.

4.2 Research and Development

Research and development ventures (R&D) can take several years to complete and they can be treated as sequential investments. For these types of projects, uncertainty usually manifests itself through the output price and the cost of completion. Because a new product, a new process or a new technology is developed, the decision maker will likely face uncertainty concerning costs, and this uncertainty can only be resolved by investing.[11] Random catastrophic events can also jeopardize the viability of the project, for example if there is a possibility that a competitor is quicker in realizing the same innovation.

All of the above characteristics considerably complicate the analysis compared to the case where only future revenues are uncertain. These difficulties do not prevent us from using the real options methodology to determine optimal investment policies for R&D ventures.[12]

As in the second model of the previous section, we suppose that a maximum rate of investment is allowed at each period. The total cost of completion for the project is now a random variable. Because of this, the minimum time needed to complete the project is now a stochastic state variable.

Furthermore, it is important to know if the patent has been granted even

[11] To simplify the discussion we do not consider input cost uncertainty. The reason for this is that its effect on investment is no different from that of cash flow uncertainty.

[12] See Schwartz (2002).

if the R&D activity is not yet complete. Because cash flows are likely to fall drastically at the expiration of a patent, the profitability of the innovation can depend entirely on the remaining length of the patent.

The first option to be exercised is the commencement option. Once the project is started, the firm has the option to abandon or slow down the investment. The most complicated case to model is when the patent is granted before the project is completed. For this case, the following simplifying assumptions are needed to obtain a solution:

1. the firm will either invest the maximum possible amount or not invest at all, and
2. if the project is abandoned it cannot be restarted later.

The second condition implies that we ignore the option to restart an idle project and we only consider the option to commence and abandon. The negative effect of this assumption is reduced by the fact that the patent is granted before the project is completed. The reduction of the patent duration during an idle period reduces the present value of the cash flows from the investment. This makes delaying an investment very costly.

The analysis yields a function that indicates for each level of investment remaining the minimum level of cash flows that justifies the continuation or the commencement of the project. Because the patent has a limited duration, the function also depends on time.

Simulation results show that standard NPV method significantly understates the value of a project. This can be a serious problem if the NPV turns out to be negative even though it is optimal to start the process. The option of abandoning in order to avoid sunk costs in an unfavorable situation is very valuable. In a multi-step investment process, the real options methodology does not only concentrate on the future cash flows related to a particular step but it also considers the importance of the future opportunities that are available by realizing an intermediate step. The standard NPV not only understates the true value of the project but also fails to provide the decision rule needed to optimally manage the opportunity during its realization.

Finally, as we mentioned at the beginning of this section, both the cash flows and the cost of completion are uncertain. The cost of completion is affected by technical uncertainty that can only be resolved by investing. Everything else being equal, this type of uncertainty creates no incentive to wait, the manager learns about his costs only by investing. Because of this, investing has a shadow value that is related to learning. This characteristic contributes to reducing the total expected cost of completion and augmenting the value of the project. An increase in uncertainty augments the value of learning and this contributes to a larger project value.

For its part, the effect of cash flow uncertainty on the value of the investment is the same as for cost uncertainty but for a different reason. More cash flow uncertainty increases the potential upside of the project leaving unchanged the downside because of the possibility to abandon the project in an unfavorable situation. Equivalently, because of the higher probability of an unfavorable situation, the abandonment option has more value.

4.3 Information Technology Investments

Information technology (IT) investments can be divided into the two following categories:

1. IT acquisition projects, and
2. IT development projects.

In an IT acquisition project, the firm has the option to spend a nonrecoverable lump sum to acquire the benefits of a specific IT asset. For the general case, we impose a time limit after which the opportunity is no longer available and no cash flows can be realized, given that technological advancements can render the actual technology obsolete. For this model, we suppose that both the acquisition cost and the benefits related to the asset are stochastic.

Purchasing the technology is equivalent to exercising an American option before maturity. The above decision process description is illustrated on Figure 5 where C represents the cash flows from the investment, K the acquisition cost, T the time limit and t the time at which it is optimal to invest.

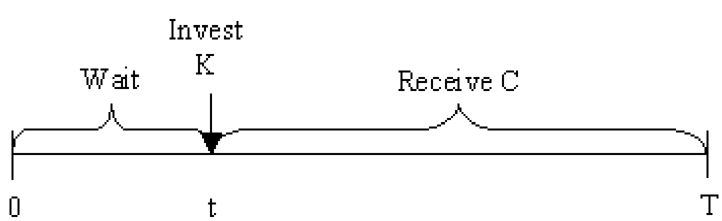

Figure 5. Decision process for an IT acquisition project.

The real options solution to the above problem yields, for every value of acquisition costs and time, the benefit level necessary for immediate investment. In addition, because of technological advances, the present value of the benefits for the current technology diminish as time goes by. As before, uncertainty and irreversibility create an incentive to wait and invest when the probability of an unfavorable situation is smaller. Consequently, the real options optimal investment threshold is superior to that of the standard NPV when waiting is optimal. Even though the NPV is positive, it may not be optimal to invest right away. The standard NPV in this case does not lead to a solution that maximizes the present value of the investment opportunity. As for the previous models, the value of the option to wait grows with the level of volatility in both state variables.

In an IT development project, the total development costs are not expensed as soon as it is optimal to start investing. The decision process in this case is very similar to an optical fiber network expansion and a R&D project. Consequently, the analysis yields a function that gives the optimal investment rule as a function of the present value of a completed project and the amount of investment required to complete. For each amount of investment required to complete the project, the threshold output price corresponds to the minimum project value that justifies investment. The contrast with the standard NPV is the same as before.

5 Possible uses of Real Options in the Telecommunications Regulation Context

A discussion of telecommunication economics will highlight the usefulness of real options in the formation of regulatory policies in the industry. Catalyzed by the Telecommunications Act of 1996 (USA), several authors have recognized the importance of developing regulatory pricing policies that take into account the new reality of the deregulated telecommunications industry. Inappropriate policies may have disastrous impacts on innovation and investment with likely consequences being a state of chronic under-investment and significant consumer surplus losses.

Currently, one of the important examples involving real options deals with prescriptions of the Local Competition and Interconnection Order of August 1996 (USA). In this case, an incumbent local exchange carrier (LEC) must give access to its local network to competitors, in particular interexchange carriers (IXC), at a reasonable price. The competitor is not required by law to engage in a long term contract. Moreover, the actual access price rule implicitly supposes that the market is perfectly contestable and that there is no demand uncertainty.

The consequence of such an access policy may be to lead to an access price that is consistent neither with the presence of demand uncertainty nor with the irreversible character of investments in telecommunications infrastructure. As expected, the deterministic perfect contestability standard leads to access prices that are lower than those obtained using the more adequate real options methodology which explicitly recognizes the additional cost component that forgone real options represent. It is therefore important to consider real options in the determination of access prices and conditions.

There is considerable debate in academic and regulatory circles regarding the proper theoretical definition and empirical computation of the "cost" on which the access price should be based. Properly addressing these questions requires proper accounting of two different but related factors: the real option approach to costing and valuing investment in infrastructures as well as the cost sharing approach emerging from cooperative game theory, a form of full cost allocation based on concepts of efficiency, incentives and equity.

6 Conclusion: The Management Information System

The real options mind-set can be characterized as the explicit recognition that uncertainty creates opportunities and value which require adequate decisions in order to materialize. It gives rigorous content to many high level managers' objectives and intuitive decision-making behavior, that is attach importance to the timing of decisions, control downside risks and capture upside opportunities, and develop and manage flexibility positions.

What does one need to promote and apply a real options approach to investment analysis? In a context of imperfect and incomplete information as well as exogenous uncertainty, a decision making process, making proper use of the best available analytical expertise, rests on a concerted search and identification of new information as it becomes available and on the efficient processing of that information.

Processing the information means its translation into the analytical language of state variable evolution (the new levels of the fundamental variables) and the state variable volatility (the new volatility levels of the fundamental variables if there are reasons to reevaluate these volatility levels). Hence, the identification of the sources of uncertainty, of the specific decisions that raise exposure to profitable outcomes and/or reduce exposure to downside risk, and the design of optimal decision rules are key ingredients a real options approach.

In many cases, the systematic gathering of new information will be complemented by a simulation capability in order to determine whether it is time to exercise an option or not, that is, whether it is time or not to make the decision to invest, expand, contract, enter, exit, abandon, etc.

In many aspects, the real options approach and mind-set apply the rigor, discipline and accuracy of finance in other decision-making areas. The approach and mind-set are relevant to a wide range of strategic decisions under uncertainty and irreversibility. Developing and implementing a real options mind-set among top level executives is nevertheless a challenging task. Analytical tools of finance must be adapted and complemented with industrial analysis and forecasting methods and moreover, each application is likely to be context specific.

The real options approach is "a capacity and willingness to detect decisions that create opportunities or protect against mishaps, and act upon them in order to create value for the firm. For managers with such a state of mind, the real options approach is a tool that allows them to bring intuition in line with the prescriptions of rigorous decision-making procedures. More importantly it allows them to give a more accurate quantitative content and value to intuitive rules, thus gaining an edge over competitors."[13]

[13] See Boyer, Christoffersen, Lasserre, Pavlov (2003).

References

[1] Alleman, James, and Eli Noam. 1999. *The New Investment Theory of Real Options and its Implication for Telecommunications Economics.* Boston, Massachusetts: Kluwer Academic Pub..

[2] Amran, Martha, and Nalin Kulatilaka. 1998. *Real Options: Managing Strategic Investment in an Uncertain World.* Boston, Massachusetts: Harvard Business School Press.

[3] Bar-Ilan, Avner, and William C. Strange. 1996. "Investment Lags." *The American Economic Review* 86 (June) : 610-622.

[4] Boyer, Marcel et Jacques Robert. 1998. "Competition and Access in Electricity Markets : ECPR, Global Price Cap and Auctions" pp. 47-74 in G. Zaccour (ed.), *Deregulation of Electric Utilities*, Kluwer Academic Pub.

[5] Boyer, Marcel, Christoffersen, Peter, Lasserre, Pierre, et Andrey Pavlov. 2003. "Value Creation, Risk Management, and Real Options" CIRANO 2003RB-01.

[6] Boyer, Marcel, Lasserre, Pierre, Mariotti, Thomas et Michel Moreaux. 2001. "Lumpy Investment, Real Options and the Dynamics of Industry Development", CIRANO 2001s-64.

[7] Boyer, Marcel, Moreaux, Michel et Michel Truchon. 2002. "Le partage des coûts communs: enjeux, problématique et pertinence", CIRANO 2002RB-03.

[8] Christoffersen, Peter and Andrey Pavlov. 2003. "Company Flexibility, the Value of Management and Managerial Compensation", CIRANO 2003s-06.

[9] Copeland, Tom, and Vladimir Antikarov. 2001. *Real Options: A Practitioner's Guide*. New York, New York: Texere.

[10] Cox, John C., Jonathan E. Ingersoll Jr., and Mark Rubenstein. 1979. "Option Pricing: A Simplified Approach." *Journal of Financial Economics* 7: 229-263.

[11] Dixit, Avinash., and R.S. Pindyck. 1994. *Investment under Uncertainty.* Princeton, New Jersey: Princeton University Press.

[12] Gamba, Andrea. 2002. "Real Options Valuation: a Monte Carlo Simulation Approach." Working Paper, Department of Financial Studies, University of Verona - Italy. 40 pages.

[13] Grenadier, Steven (Editor). 2000. *Game Choices: The Intersection of Real Options and Game Theory*. London, England: Risk Books.

[14] Huisman, Kuno J.. 2001. *Technology Investment: A Game Theoretic Real Options Approach*. Boston, Massachusetts: Kluwer Academic Pub..

[15] Hull, John. 1997. *Options, Futures, and Other Derivatives.* Upper Saddle River, NJ: Prentice-Hall.

[16] Lassila, Janne. 2001. "Real Options in Telecommunications Capacity Markets." Helsinki University of Technology, Department of Engineering Physics and Mathematics. 79 pages.

[17] Longstaff, Francis A., and Eduardo S. Schwartz. 2001. "Valuing American Options by Simulation: A Simple Least-Squares Approach." *The Review of Financial Studies* 14 (Spring): 113-147.

[18] Majd, Saman, and Robert S. Pindyck. 1987. "Time to Build, Option Value, and Investment Decisions." *Journal of Financial Economics* 19 (March) : 7-27.

[19] Milne, Alistair, and A. Elizabeth Whalley. 2000. "Time to Build, Option Value, and Investment Decisions : A Comment." *Journal of Financial Economics* 56 (May) : 325-332.

[20] Schwartz, Eduardo S. and Carlos Zozaya-Gorostiza. 2000. "Valuation of Information Technology Investments as Real Options." 4th Annual International Conference on Real Options.

[21] Schwartz, Eduardo S.. 2002. "Patents and R&D as Real Options." Working Paper, UCLA. 49 pages.

A Analytical Techniques

This appendix is a summary of the techniques and the hypotheses used to develop a real options model for investment decisions. However, before we present this framework, we will review the concepts of continuous time compounding and continuous time present values.

A.1 The Continuous Time Framework

In all of the following models we will work with a continuous time framework. In this setting it is supposed that:

1. decisions and transactions can be made at each instant,
2. interest is compounded continuously,
3. stochastic variables can change at any time, and
4. revenues from a project are realized at each instant.

In this section, we will show how to handle discounting when the revenues from a project are realized at each instant (continuously). First of all, we define the limit

$$\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = e. \tag{A.1}$$

If we let m equal the number of compounding periods in one year and r the interest rate, the present value of a dollar received in t periods from now when interest is compounded m times per year is equal to

$$PV = \left(1 + \frac{r}{m}\right)^{-mt}. \tag{A.2}$$

If we define $w = \frac{m}{r}$, we can transform (A.2) into the alternative form

$$PV = \left[\left(1 + \frac{1}{w} \right)^{w} \right]^{-rt}. \tag{A.3}$$

Using (A.1), we take the limit of (A.3) when $m \to \infty$ ($m \to \infty \Rightarrow w \to \infty$) we get

$$PV = e^{-rt}. \tag{A.4}$$

Equation (A.4) is equivalent to the value of a dollar received in t periods when interest is compounded continuously ($m \to \infty$). If we define $P(t)$ as the continuous rate of yearly cash flows for a project at instant t, the discounted value of these cash flows (DCF) up to time T when cash flows are realized m times per year is equal to

$$DCF = \sum_{i=1}^{mT} \frac{1}{m} P\left(\frac{i}{m} \right) \left[\left(1 + \frac{r}{m} \right)^{m} \right]^{-\frac{i}{m}}. \tag{A.5}$$

Because the expression in (A.5) is continuous, taking the limit of (A.5) as $m \to \infty$ is equivalent to supposing that cash flows are realized continuously, by taking the limit we obtain

$$\lim_{m\to\infty} DCF = \lim_{m\to\infty} \sum_{i=1}^{mT} \frac{1}{m} P\left(\frac{i}{m} \right) \left[\left(1 + \frac{r}{m} \right)^{m} \right]^{-\frac{i}{m}} = \int_{0}^{T} P(t)\, e^{-rt} dt, \tag{A.6}$$

where $dt = \lim_{m\to\infty} \frac{1}{m}$, $t \in (0, 1]$ and (A.3) and (A.4) are used. Consequently, when cash flows are realized continuously, the integral replaces the sum when it comes to computing present values. For example, if we have $P(t) = D$, (A.6) becomes

$$D \int_{0}^{T} e^{-rt} dt = -\frac{D}{r} \left[e^{-rt} \right]_{0}^{T} = \frac{D}{r} \left(1 - e^{-rT} \right).$$

A.2 Ito Processes and Ito's Lemma

In real options analysis we deal with functions of stochastic variables. Usually, we suppose that these stochastic variables evolve according to Ito processes. Ito processes are used because they are rich enough to represent many observed stochastic phenomena and they have the advantage of being relatively easy to work with. An Ito process can be characterized by

$$dx = a(x, t)\, dt + b(x, t)\, dz. \tag{A.7}$$

Here dz is referred to as a standard Wiener process with $dz = \varepsilon_t \sqrt{dt}$ and $\varepsilon_t \sim N(0, 1)$. The Ito process most common to real options analysis is the geometric Brownian motion

$$dx = \alpha x dt + \sigma x dz. \tag{A.8}$$

For example, if $dt = 1$ (one year), $\alpha = 0,02$ (in annual terms) and $\sigma = 0,20$, we have for (A.8)

$$\frac{dx}{x} = a + \sigma\varepsilon_t.$$

According to the above equation, the return on x, that is $\frac{dx}{x}$, for a one year period is equal to the expected return α plus an unexpected perturbation $\sigma\varepsilon_t$, with $\varepsilon_t \sim N(0, 1)$. Two realizations of a geometric Brownian motion for x with $dt = \frac{1}{365}$ (one day), α and σ as above are illustrated on the graph below.

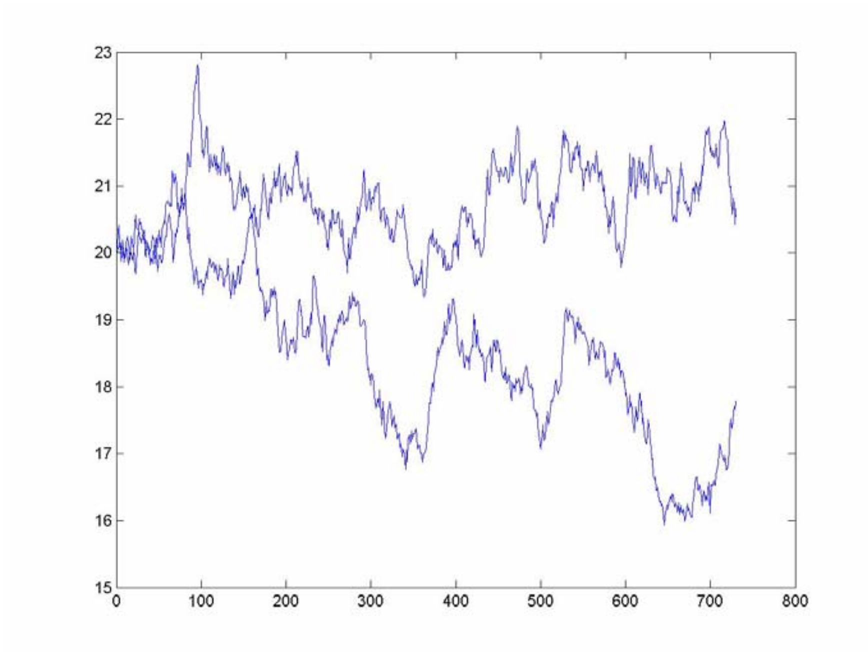

Figure A.1. Sample paths for a variable x that evolves according to a geometric Brownian motion with $\alpha > 0$.

Even though the above process is continuous, the standard rules of calculus cannot be applied to functions of x because of the "irregular" behavior of x. Let us write the value of an asset or an investment opportunity as $F(x,t)$. For real options, we first must find how the value of the opportunity varies with x, and because of this, the most needed operation is the differential dF. If x follows an Ito process, Ito's Lemma states that the differential for functions of x takes the form:

$$dF = \left[\frac{\partial F}{\partial t} + a(x,t)\frac{\partial F}{\partial x} + \frac{1}{2}b^2(x,t)\frac{\partial^2 F}{\partial x^2}\right]dt + b(x,t)\frac{\partial F}{\partial x}dz. \qquad \text{(A.9)}$$

Finally, if F is a function of several possibly correlated variables, $x_1, ..., x_n$

that follow Ito processes, the multivariate equivalent of (A.9) is

$$\begin{aligned} dF &= \left[\frac{\partial F}{\partial t} + \sum_{i=1}^{n} a_i \left(x_1, ..., t \right) \frac{\partial F}{\partial x_i} \right. \\ &\left. + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_i \left(x_1, ..., t \right) b_i \left(x_1, ..., t \right) \rho_{ij} \frac{\partial^2 F}{\partial x_i \partial x_i} \right] dt \\ &+ \sum_{i=1}^{n} b_i \left(x_1, ..., t \right) \frac{\partial F}{\partial x_i} dz_i. \end{aligned} \quad \text{(A.10)}$$

Where ρ_{ij} is the correlation coefficient between two processes with $\rho_{ii} = \sigma_i^2$.

A.3 Stochastic Dynamic Programming

For dynamic programming, the value of an entire decision sequence is split-up into two components. These components are:

1. the value related to an immediate decision, and
2. a function that reflects the value of all subsequent decisions conditional on the immediate one.

The value of the second component is optimized with respect to all subsequent decisions. Consequently, the sum of both elements needs only to be maximized with respect to the immediate decision.

The most frequent class of applications in real options analysis is called optimal stopping problems. Here $F(x_t, t)$ denotes the value of an optimally managed investment opportunity. We suppose here that x_t follows an Ito process and that the value of the opportunity can depend on time. At each instant, the decision maker must choose between:

1. invest to obtain the payoff of the investment $\Omega(x_t, t)$, or
2. defer the decision to the next period.[14]

This problem expressed in its dynamic programming form and in discrete time can be written as:

$$F(x_t, t) = \max\left\{\Omega(x_t, t),\ \pi(x_t, t) + \frac{1}{1+\rho}E\left[F(x_{t+1}, t+1) \mid x_t\right]\right\}. \quad \text{(A.11)}$$

Where $\pi(x_t, t)$ is a possible profit flow related to waiting and ρ is the appropriate discount rate. Here $E\left[F(x_{t+1}, t+1) \mid x_t\right]$ is the expected value of the next period's optimal value function conditional on not investing today. This is equivalent to the expected value of the opportunity when the decision is optimally taken at a $(t + n^{th})$ period. For some x_t's, the maximum on the right hand side of (A.11) will be achieved by choosing $\Omega(x_t, t)$. For the cases that concern us, only one value of x_t denoted by $x^*(t)$ marks the boundary between waiting and investing.

Before we continue, a distinction must be made between a time dependant and an autonomous problem. For a time dependant problem, the decision maker does not have the luxury of delaying indefinitely. Eventually at some date T, he will be in a "take it or leave it" situation and (A.11) will become

$$F(x_T, T) = \Omega(x_T, T). \quad \text{(A.12)}$$

In this case, the threshold that marks the boundary between waiting and investing is a function of time $x^*(t)$. For an autonomous problem, the decision maker is not constrained by time and he can delay the investment indefinitely. The threshold x^* in this situation is independent of time.

[14] Where in this case, $\Omega(x_t, t)$ usually takes the form $\max[\cdot,\ 0]$.

Usually for problems of this type, a continuous time framework is used. Each period has a length of Δt and we are interested in the limit problem as $\Delta t \to 0$. In this case, it is supposed that the decision maker can act at each instant. For an optimal stopping investment problem when it is optimal to wait (waiting region), (A.11) is equal to

$$F(x_t, t) = \pi(x_t, t)\,\Delta t + \frac{1}{1+\rho\Delta t} E\left[F(x+\Delta x, t+\Delta t) \mid x_t\right]. \tag{A.13}$$

If we multiply (A.13) by $(1+\rho\Delta t)$ and rearrange we have

$$F(x_t, t)\,\rho\Delta t = \pi(x_t, t)\,\Delta t\,(1+\rho\Delta t) + E[\Delta F]. \tag{A.14}$$

Finally, if we divide (A.14) by Δt and let $\Delta t \to 0$ we get

$$\rho F(x_t, t) = \pi(x_t, t) + \frac{1}{dt} E[dF]. \tag{A.15}$$

According to (A.15), in the waiting region, an optimally managed investment opportunity is equivalent to an asset with a value of $F(x_t, t)$. The normal return on this asset $\rho F(x_t, t)$ must equal the immediate payoff $\pi(x_t, t)$ plus the expected capital gain $\frac{1}{dt}E[dF]$. If x_t evolves according to (A.7), we can apply Ito's Lemma to (A.15) and show that $F(x_t, t)$ must satisfy the partial differential equation

$$\begin{aligned} \rho F(x_t, t) &= \pi(x_t, t) + \frac{1}{dt}\left\{\left[F_t(x_t, t) + a(x,t)\,F_x(x_t, t) \right.\right.\\ &\qquad \left.\left. + \frac{1}{2}b^2(x,t)F_{xx}(x,t)\right] dt\right\} \\ &= \pi(x_t, t) + F_t(x_t, t) + a(x,t)\,F_x(x_t, t) \\ &\qquad + \frac{1}{2}b^2(x,t)F_{xx}(x,t). \end{aligned} \tag{A.16}$$

As mentioned, the threshold $x^*(t)$ marks the boundary between waiting and investing. At $x^*(t)$ the firm is indifferent between the two alternatives, we thus have the boundary condition (value matching condition)

$$F(x^*(t), t) = \Omega(x^*(t), t). \tag{A.17}$$

For this type of problem, $x^*(t)$ is endogenous and it must be found along with $F(x_t, t)$, we thus need the extra boundary condition

$$F_x(x^*(t), t) = \Omega_x(x^*(t), t). \tag{A.18}$$

Condition (A.18) is called a smooth-pasting condition. It requires that the slopes of both functions match at the boundary. If the problem is time dependent, we must add the terminal condition (A.12) and in both cases we need a condition at $x = 0$. Finally, if the problem is autonomous, we have $F_t(x_t, t) \equiv 0$, and we can write $F_x(x_t, t) = F'(x)$ and $F_{xx}(x, t) = F''(x)$.

An example of an autonomous optimal stopping problem will clarify the above explanations.[15] In this example, the decision maker can pay a sunk cost I in return for a project whose value is given by $F(V)$, where V evolves according to the geometric Brownian motion

$$dV = \alpha V dt + \sigma V dz, \tag{A.19}$$

where dz is the increment of a standard Wiener process. The goal is to determine the value of the investment opportunity denoted by $F(V)$ and the critical value V^* at which it is optimal to incur the sunk cost I.[16] Because no cash flows are realized by simply holding the opportunity, we suppose that $\pi(x_t, t) = 0$. According to (A.15), in the waiting region $F(V)$ must satisfy

$$\rho F(V) = \frac{1}{dt} E[dF]. \tag{A.20}$$

With the help of Ito's Lemma, we expand (A.20) to get

$$\begin{aligned} \rho F(V) &= \frac{1}{dt} E\left[\alpha V F'(V) dt + \frac{1}{2}\sigma^2 V^2 F''(V) dt + \sigma V F'(V) dz\right] \\ &= \alpha V F'(V) + \frac{1}{2}\sigma^2 V^2 F''(V) \quad (E[dz] = 0), \end{aligned} \tag{A.21}$$

[15] See Dixit and Pindyck (1994) for a more detailed exposition.

[16] For this example, we consider an autonomous problem.

that is

$$\alpha V F^{'}(V)+\frac{1}{2}\sigma^{2}V^{2}F^{''}(V)-\rho F(V)=0. \tag{A.22}$$

With the boundary conditions

$$F(0)=0 \tag{A.23}$$

$$F(V^{*})=V^{*}-I \tag{A.24}$$

$$F^{'}(V^{*})=1. \tag{A.25}$$

Conditions (A.24) and (A.25) are the continuity or value matching condition and the smooth pasting condition. According to (A.23), if V is ever equal to zero, it remains there indefinitely, the opportunity is then worthless. The general solution to equation (A.22) is

$$F(V)=A_{1}V^{\beta_{1}}+A_{2}V^{\beta_{2}} \tag{A.26}$$

where A_1 and A_2 are constants to be determined and β_1 and β_2 are the roots of the quadratic equation

$$\frac{1}{2}\sigma^{2}\beta(\beta-1)+\alpha\beta-\rho=0. \tag{A.27}$$

We have, $\beta_1 > 1$ and $\beta_2 < 0$. Because of condition (A.23) and $\beta_2 < 0$ we must have $A_2 = 0$, otherwise $F(V)$ will tend to infinity when V goes to zero. Finally, (A.24) and (A.25) can be used to find A_1 and V^*:

$$V^{*}=\frac{\beta_{1}}{\beta_{1}-1}I \quad \text{with} \quad \frac{\beta_{1}}{\beta_{1}-1}>1 \tag{A.28}$$

and

$$A_{1}=\frac{(V^{*}-I)}{(V^{*})^{\beta_{1}}}. \tag{A.29}$$

These results are consistent with those of the simple two period example presented previously. According to (A.28) and because $\beta_1 > 1$, with uncertainty, the optimal investment criterion requires that $V^* > I$ while NPV requires $V^* = I$. The graphic below illustrates the solution $F(V)$ and the straight line represents $\max[V - I,\ 0]$.

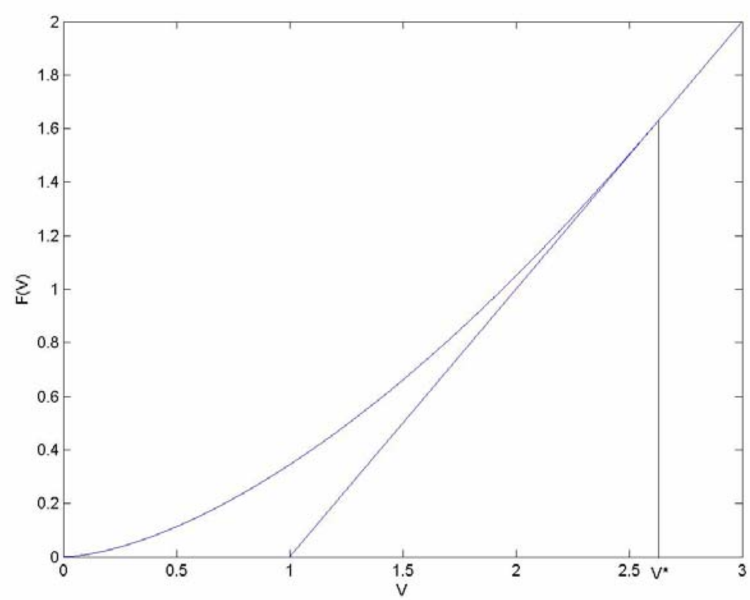

Figure A.2. Graphic of solution to optimal stopping example.

A.4 Contingent Claims Analysis

The essence of contingent claims analysis is to use specific combinations of traded assets to determine the value of a non-traded asset. In this framework, one assumes that it is possible to create portfolios that exactly replicate the risk and return characteristics of a non-traded asset. Two assets that have the same risk and return characteristics must have the same value, otherwise arbitrage opportunities would exist. If we rule out arbitrage opportunities, contingent claims analysis allows us to find the value of the non-traded asset.

If the asset is an investment opportunity, the optimal investment strategy can also be found. The following example illustrates the aforementioned principles.

As in the previous section, $F(x_t, t)$ represents the value of an investment opportunity and x_t the firm's output. To apply contingent claims analysis, the output must either be directly traded on organized markets or spanned by other traded assets.[17] To simplify the example, we suppose that x_t is traded and that it follows the geometric Brownian motion

$$dx = \alpha x dt + \sigma x dz. \tag{A.30}$$

The asset's average growth rate is equal to α and we define μ as the expected risk adjusted expected return required to hold x.[18] Furthermore, we suppose that $\mu > \alpha$ and that $\delta = \mu - \alpha$ represents the dividend or convenience yield related to holding x. Consequently, producing and selling a unit of x is not the same as holding the opportunity to produce it, there is an opportunity cost related to holding and not producing that is equivalent to δ. To build the replicating portfolio, we invest one dollar in the riskless asset (with return r) and we purchase n units of x, this portfolio costs $(1 + nx)$ dollars. For a short period of time dt, the return on the riskless asset is rdt and the random return on x is (A.30) plus the dividend or convenience yield. Consequently, the random return per dollar invested in the portfolio is equal to

$$\frac{r + n(\alpha + \delta)x}{1 + nx} dt + \frac{\sigma n x}{1 + nx} dz. \tag{A.31}$$

In the waiting region, the return from holding the investment opportunity arises strictly from capital gains because we suppose that no other revenues

[17]A spanning asset must have the same uncertainty profile as the asset to be spanned.
[18]We could use the CAPM to determine μ.

are realized from simply holding the opportunity. By Ito's Lemma, the random capital gain for the investment opportunity is

$$\begin{aligned} dF &= \left[F_t(x_t,t) + \alpha x F_x(x_t,t) + \frac{1}{2}\sigma^2 x^2 F_{xx}(x_t,t)\right] dt \\ &\quad + \sigma x F_x(x_t,t)\, dz, \end{aligned} \tag{A.32}$$

and the random per dollar return for the investment opportunity is equal to

$$\frac{\left[F_t(x_t,t) + \alpha x F_x(x_t,t) + \frac{1}{2}\sigma^2 x^2 F_{xx}(x_t,t)\right]}{F(x_t,t)} dt + \frac{\sigma x F_x(x_t,t)}{F(x_t,t)} dz. \tag{A.33}$$

The second parts of (A.31) and (A.33) represent the risky components of the returns on both assets. If we want the portfolio to exactly replicate the risk of owning the investment opportunity, we must have

$$\frac{nx}{1+nx} = \frac{xF_x(x_t,t)}{F(x_t,t)}. \tag{A.34}$$

If we have (A.34), both assets have the same risk. Consequently, the expected per dollar return from holding the portfolio must be equal to that of holding the opportunity, in this case

$$\frac{F_t(x_t,t) + \alpha x F_x(x_t,t) + \frac{1}{2}\sigma^2 x^2 F_{xx}(x_t,t)}{F(x_t,t)} = \frac{r + n(\alpha+\delta)x}{1+nx}. \tag{A.35}$$

If we substitute (A.34) in (A.35), we show that $F(x_t,t)$ must satisfy

$$\frac{1}{2}\sigma^2 x^2 F_{xx}(x_t,t) + (r-\delta) x F_x(x_t,t) + F_t(x_t,t) - rF(x_t,t) = 0, \tag{A.36}$$

along with the boundary conditions that were specified in the previous section.

Finally, if the state variable is not a traded asset, a traded asset that spans the uncertainty of the state variable can be used to form the replicating portfolio. Contrary to the dynamic programming approach, no hypotheses are needed concerning the discount rate and the growth rate of the state variable. However, it is necessary to have assets that span the risk and return characteristics of the asset to be valued.

A.5 Equivalent Martingale Measures

In the preceding section, no arbitrage arguments are used to obtain a partial differential equation that enables us to find the project's valuation formula, explicitly or numerically. With the same assumptions, another equivalent solution method is available.

With the equivalent martingale measure method, the probability distribution of the state variable discounted at the risk free interest rate is transformed into a martingale.[19] In this case, the current value of the investment opportunity is equal to its expected (with the transformed distribution) future value discounted at the risk free rate. It can be shown that the valuation function obtained with the equivalent Martingale measure method satisfies the same partial differential equation as the one obtained with contingent claims analysis.

[19] Simply put, a stochastic process is a martingale if the best forecast of an unobserved future value is equal to the last observed value.

B Solution Methods

The most common solution methods for real options problems will be presented in this section. Our goal is not to provide an exhaustive description of the methods. We aim to show the reader that well structured real option problems are not simply abstract descriptions of reality, but can be resolved and fruitfully applied to real life situations.

B.1 The Binomial Method

We illustrate the binomial method with the help of a simple American option valuation problem.[20] For this method, we assume that in a short interval of time the value of the state variable can only experience a specified up or down movement.

To understand the intuition behind the method, we first value a European call option. This option has an exercise price of $21 and expires in three months. We suppose that the stock pays no dividends. Furthermore, in three months the stock can take only two values, $22 or $18. The above situation is illustrated in figure B.1.

[20] See Hull (1997) and Cox, Ross and Rubinstein (1979).

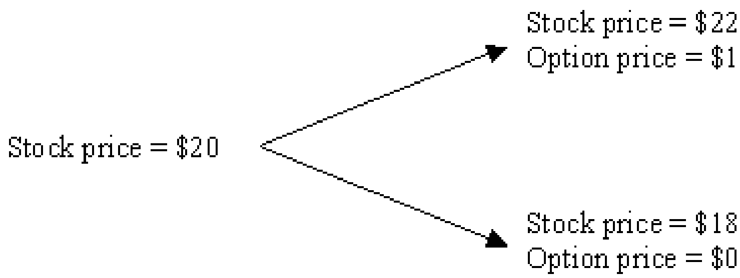

Figure B.1. Evolution of the underlying stock price for the one period European option example.

The idea behind the binomial method is to evoke the no arbitrage condition to obtain a valuation formula. In this case, the value of the option will be independent of subjective information such as the investor's risk preferences and the probability of an up or down movement in the stock price.

To find the value of the aforementioned European option, we consider a portfolio composed of a long position in Δ shares of the underlying stock and a short position in one call option. We choose Δ so that portfolio has the same value in all of the contingencies illustrated in figure B.1, that is:

$$22\Delta - 1 = 18\Delta \;\Rightarrow\; \Delta = 0.25.$$

Consequently, the portfolio composed of a long position in 0.25 shares of the underlying stock and a short position in one call option will then be worth $22 \times 0.25 - 1\ \ (0.25 \times 18) = 4.5$ in both contingencies. The above portfolio is then risk free. Because of this, it must earn the risk free rate of return. If this is not the case, arbitrage opportunities are available and this violates the presupposed no arbitrage condition.

We suppose here that the risk free interest rate is equal to 12%, the portfolio at the beginning of the period must then be worth $4.5e^{\frac{-0.12}{4}} = 4.367$. The price of the option today is denoted by f and it is equal to

$$20 \times 0.25 - f = 4.367 \;\Rightarrow\; f = \$0.633.$$

We can generalize the above one period example. Suppose that the current stock price S can move up to Su or down to Sd $(u > 1,\ d < 1)$. The returns in both situations are $u-1$ and $d-1$. The payoff of the derivative after an up or down movement is equal to f_u and f_d respectively (f_u and f_d are supposed known at the beginning of the period), and the current value of the derivative is denoted by f.

As before, we choose a long position in Δ shares so that the portfolio composed of a short position in the option and a long position in the stock is riskless, we must have

$$Su\Delta - f_u = Sd\Delta - f_d \;\Rightarrow\; \Delta = \frac{f_u - f_d}{Su - Sd}.$$

The cost of the portfolio must then be equal to

$$S\Delta - f = (Su\Delta - f_u)\, e^{-rT},$$

where $r > 0$ is the risk free rate and T is the length of period. If we substitute for Δ we get

$$f = e^{-rT}\left[p f_u + (1-p) f_d\right], \tag{B.1}$$

where

$$p = \frac{e^{rT} - d}{u - d}. \tag{B.2}$$

An interesting interpretation can be given to expressions (B.1) and (B.2). Because we suppose that $d - 1 < r < u - 1$, (B.2) can be interpreted as a probability.[21] If we suppose that q is the probability of an up movement in the price of the stock, the expected value of the stock price in a binomial model is

$$E\left(S_T\right) = qSu + (1-q)\, Sd. \tag{B.3}$$

If we substitute (B.2) for q in (B.3) we get

$$E\left(S_T\right) = Se^{rT}.$$

Hence, p is equivalent to the probability of an up movement in a risk neutral world. Consequently, according to (B.1), using the binomial valuation method is equivalent to supposing a risk neutral world (with its corresponding probability distribution) to price an option.

The above example can easily be extended to an option with early exercise features (American option). A two period example is summarized in figure B.2. In this case, $P\left(S\right)$ is a general payoff function that depends on the stock price, T is the length of a period and $f_{u^i d^{n-i}}$ is the value of the option at each node with

$$\begin{aligned} n &= 0 \text{ and } i = 0 \text{ at the initial node,} \\ n &= 1 \text{ and } i = 0, 1 \text{ after the first period, and} \\ n &= 2 \text{ and } i = 0, 1, 2 \text{ after the second period.} \end{aligned}$$

[21] If we have $d-1 < u-1 < r$ a riskless profit can be made by shorting the stock and lending at the risk free rate. Under the assumption of no arbitrage we can rule out this possibility. The situation $r < d-1 < u-1$ is not possible according to the assumptions on r $(r > 0)$ and d $(d-1 < 0)$.

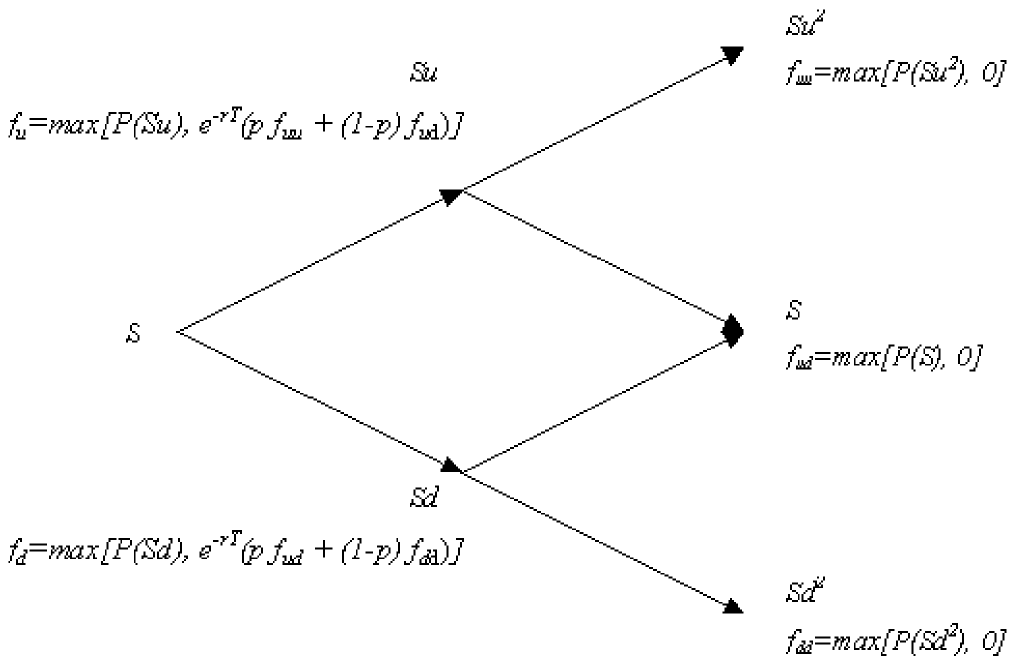

Figure B.2. Generalization of the Binomial Method for an American option

At each node, the holder of the option must decide if it is more profitable to exercise now or to wait and exercise at a latter period. Consequently, the value of the option at each node is equal to the maximum between the value of the payoff function and the expected continuation value.

At the terminal node (after the second period) we have

$$\begin{aligned} f_{uu} &= \max\left[P\left(Su^2\right),0\right] \quad \text{if the price reaches } Su^2 \\ f_{ud} &= \max\left[P\left(S\right),0\right] \quad \text{if the price remains at } S \\ f_{dd} &= \max\left[P\left(Sd^2\right),0\right] \quad \text{if the price falls to } Sd^2, \end{aligned}$$

and at the intermediate node (after the first period) we have

$$\begin{aligned} f_u &= \max\left[P\left(Su\right), e^{-rT}\left(pf_{uu} + (1-p)\, f_{ud}\right)\right] \quad \text{if the price reaches } Su \\ f_d &= \max\left[P\left(Sd\right), e^{-rT}\left(pf_{ud} + (1-p)\, f_{dd}\right)\right] \quad \text{if the price falls to } Sd. \end{aligned}$$

Finally, the value of the option at the initial node is then given by

$$f = \max\left[P\left(S\right),\ e^{-rT}\left(pf_u + (1-p)\, f_d\right)\right].$$

B.2 The Finite Difference Method

The dynamic programming and contingent claims methods yield partial differential equations with boundary and initial conditions. If no closed form solution exist for these partial differential equations, numerical techniques have to be employed. One of the most common techniques is the finite difference method.

A general example similar to a problem often encountered in real options analysis will be used to describe the finite difference method. In this case, we need to find a function $u(z,\tau)$ that satisfies the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial z^2} \tag{B.4}$$

for all z inferior to an unknown $z^*(\tau)$. Here $z \in \Re$, $\tau \in (0, \Upsilon]$ and when $z \geq z^*(\tau)$, $u(z,\tau) = h(z,\tau)$. The function $u(z,\tau)$ must also satisfy the conditions

$$u(z,0) = h(z,0), \tag{B.5}$$

$$\lim_{z \to -\infty} u(z,\tau) = 0, \tag{B.6}$$

$$u\left(z^*(\tau), \tau\right) = h\left(z^*(\tau), \tau\right), \tag{B.7}$$

$$\frac{\partial}{\partial z} u\left(z^*\left(\tau\right), \tau\right) = \frac{\partial}{\partial z} h\left(z^*\left(\tau\right), \tau\right) \tag{B.8}$$

and the constraint

$$u\left(z, \tau\right) \geq h\left(z, \tau\right). \tag{B.9}$$

In this case, $z^*\left(\tau\right)$ is an unknown boundary value that is function of τ. The aforementioned boundary value is analogous to an optimal investment threshold in a time dependant real options problem. For its part, $h\left(z, \tau\right)$ is an arbitrary function similar to the net present value of an investment. The presence of the moving boundary conditions (B.7) and (B.8) prevents us form obtaining a closed form solution to $u\left(z, \tau\right)$.

To eliminate the complications that arise from conditions (B.7) and (B.8), we express the above problem in the following linear complementary form:

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial z^2}\right) \geq 0, \quad \left(u\left(z, \tau\right) - h\left(z, \tau\right)\right) \geq 0$$

with

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial z^2}\right) \cdot \left(u\left(z, \tau\right) - h\left(z, \tau\right)\right) = 0, \tag{B.10}$$

with u and $\frac{\partial u}{\partial z}$ continuous. Both of the equalities in (B.10) hold at $z^*\left(\tau\right)$.

The next step is to formulate the problem so that it can be solved numerically. For this, we first divide the z and τ axes into equally spaced nodes that are separated by a distance of δz and $\delta \tau$ respectively. From now on, only the values of u evaluated at each node will are important and we express u as

$$u_n^m = u\left(n\delta z, m\delta \tau\right) \tag{B.11}$$

where

$$-N \leq n \leq N \text{ and } 0 \leq m \leq M.$$

Where N is a large positive integer and $M = \frac{\Upsilon}{\delta \tau}$.

The second step consists of obtaining discrete approximations for the partial derivatives in (B.4). To obtain an approximation for $\frac{\partial u}{\partial \tau}$, we take a first-order Taylor expansion with respect to τ around (z, τ)

$$u(z, \tau + \delta\tau) = u(z, \tau) + \delta\tau \frac{\partial u}{\partial \tau}(z, \tau) + \frac{1}{2}(\delta\tau)^2 \frac{\partial^2 u}{\partial \tau^2}(z, \tau + \psi\delta\tau), \quad \text{(B.12)}$$

and we use the approximation

$$\frac{\partial u}{\partial \tau}(z, \tau) \approx \frac{u(z, \tau + \delta\tau) - u(z, \tau)}{\delta\tau} = \frac{u_n^{m+1} - u_n^m}{\delta\tau}. \quad \text{(B.13)}$$

The above approximation is $O(\delta\tau)$, $-\delta\tau$ can also be used. Now, to obtain $\frac{\partial^2 u}{\partial z^2}$ we define the following Taylor expansions:

$$\begin{aligned} u(z + \delta z, \tau) &= u(z, \tau) + \delta z \frac{\partial u}{\partial z}(z, \tau) + \frac{1}{2}(\delta z)^2 \frac{\partial^2 u}{\partial z^2}(z, \tau) \\ &\quad + \frac{1}{6}(\delta z)^3 \frac{\partial^3 u}{\partial z^3}(z, \tau) + \frac{1}{24}(\delta z)^4 \frac{\partial^4 u}{\partial z^4}(z + \varepsilon\delta z, \tau) \quad \text{(B.14)} \end{aligned}$$

and

$$\begin{aligned} u(z - \delta z, \tau) &= u(z, \tau) - \delta z \frac{\partial u}{\partial z}(z, \tau) + \frac{1}{2}(\delta z)^2 \frac{\partial^2 u}{\partial z^2}(z, \tau) \\ &\quad - \frac{1}{6}(\delta z)^3 \frac{\partial^3 u}{\partial z^3}(z, \tau) + \frac{1}{24}(\delta z)^4 \frac{\partial^4 u}{\partial z^4}(z + \varsigma\delta z, \tau) . \quad \text{(B.15)} \end{aligned}$$

we add (7) and (7) to obtain the approximation

$$\begin{aligned} \frac{\partial^2 u}{\partial z^2}(z, \tau) &\approx \frac{u(z + \delta z, \tau) + 2u(z, \tau) + u(z - \delta z, \tau)}{(\delta z)^2} \\ &= \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta z)^2}. \quad \text{(B.16)} \end{aligned}$$

The above approximation is $O\left((\delta z)^2\right)$, because of this, approximations (B.13) and (7) become more accurate as the nodes get closer. With the help of expressions (B.13) and (7), we represent (B.4) in its discrete form as

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} = \frac{1}{2}\left(\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta z)^2} + \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta z)^2}\right). \quad \text{(B.17)}$$

If we define

$$\theta = \frac{\delta\tau}{(\delta z)^2}$$

and

$$W_n^m = u_n^m (1-\theta) + \frac{\theta}{2}\left(u_{n+1}^m + u_{n-1}^m\right)$$

problem (B.10) can be written as

$$u_n^{m+1}(1+\theta) - \frac{\theta}{2}\left(u_{n+1}^{m+1} + u_{n-1}^{m+1}\right) - W_n^m \geq 0$$

with

$$\left(u_n^{m+1}(1+\theta) - \frac{\theta}{2}\left(u_{n+1}^{m+1} + u_{n-1}^{m+1}\right) - W_n^m\right) \cdot \left(u_n^m - h_n^m\right) = 0. \tag{B.18}$$

Condition (B.5) in its discrete form is equivalent to $u_n^0 = h_n^0$ and (B.6) is equivalent to $u_{-N}^m = 0$.

To obtain an approximation of u_n^m for $-N \leq n \leq N$ and $1 \leq m \leq M$, an iterative algorithm is used. We shall give a brief description of the Projected Successive Over-relaxation algorithm (PSOR). Starting with the function's initial values defined by (B.5) and (B.6), the algorithm generates approximations to the unknown function values. In this case, we define $\mathbf{u^{m+1}}$ as the u_n^{m+1} vector for all m and n and k as the k^{th} iteration, the problem defined by (B.18) is solved by iterating on the following equations:

$$\begin{aligned} Y_n^{m+1,k+1} &= \frac{1}{1+\theta}\left(W_n^m + \frac{\theta}{2}\left(u_{n+1}^{m+1,k} + u_{n-1}^{m+1,k+1}\right)\right) \\ u_n^{m+1,k+1} &= \max\left(u_n^{m+1,k} + w\left(Y_n^{m+1,k+1} - u_n^{m+1,k}\right), h_n^{m+1}\right), \end{aligned} \tag{B.19}$$

with $0 < w < 2$. When $\left\|\mathbf{u^{m+1,k+1}} - \mathbf{u^{m+1,k}}\right\|_2$ (convergence criterion) is considered negligible we fix $\mathbf{u^{m+1}} = \mathbf{u^{m+1,k+1}}$. This algorithm is constructed in such a way that each constraint in (B.18) is satisfied and it is always stable and convergent.

B.3 The Least Squares Monte Carlo (LSM) Method

Before the LSM approach, Monte Carlo methods were difficult to apply to value options with early exercise features. The main difficulty came from determining the expected continuation value each time exercising the option is considered. The LSM approach ingenuously solves this problem and, in addition to this, it is flexible enough to incorporate several possibly correlated state variables.[22]

We illustrate the method with the help of a simple non-autonomous optimal stopping problem. In this case, the value of an investment opportunity depends on a single stochastic state variable P_t. In continuous time, the dynamics of the state variable are given by

$$dP = \alpha P dt + \sigma P dz, \tag{B.20}$$

with $dz = \varepsilon_t \sqrt{dt}$ and $\varepsilon_t \sim N(0,1)$. At each instant $t \in [0,T]$, the firm can either exchange the investment opportunity for a project worth $\Omega(P_t,t)$ or postpone the decision to the next period. At time t, the value of the optimally managed investment opportunity is represented by

$$F(P_t,t) = \max_{\tau \in [t,T]} \left\{ e^{-r(\tau - t)} E_t \left[\Omega(P_\tau, \tau) \right] \right\}. \tag{B.21}$$

Here τ is the optimal stopping time chosen from $[t,T]$ and E_t is the expectation conditional on the information available at time t.

To implement the method, we first divide the problem's time frame into N nodes separated by a distance of $\Delta t = \frac{T}{N}$. We are now only concerned by the following set of possible stopping times

$$\{t_0 = 0, t_1 = \Delta t, ..., t_N = N\Delta t\}.$$

[22] For more details, see Longstaff and Schwartz (2001) and Gamba (2002).

Because of this, the method yields a discrete time approximation of $F(P_t, t)$. The next step is to simulate K paths of the state variable according to the following solution of (B.20)

$$P(t + \Delta t) = P(t)\, e^{\left(\alpha - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t}. \quad \text{(B.22)}$$

We denote $P_t(w)$ as the value of the state variable along the w^{th} path at time t and $\tau(w)$ as the path's optimal stopping time. If we define t as the current date, $\tau(w)$ is generated with the information available up to date t, if P_t is Markov, it contains all this information.[23]

We proceed recursively to find (B.21) along each path. At each node, the decision maker must choose between investing and waiting. At time $N\Delta t$, the value of the opportunity is

$$F(P_{t_N}(w), t_N) = \max\{\Omega(P_{t_N}(w), t_N), 0\} \quad w = 1, ..., K, \quad \text{(B.23)}$$

and at $(N-1)\Delta t$ it is

$$\begin{aligned} F\left(P_{t_{N-1}}(w), t_{N-1}\right) &= \max\left\{\Omega\left(P_{t_{N-1}}(w), t_{N-1}\right),\right. \\ &\qquad \left. e^{-r(\Delta t)} E_{t_{N-1}}\left[F\left(P_{t_N}(w), t_N\right)\right]\right\}. \end{aligned} \quad \text{(B.24)}$$

We proceed in this fashion up to t_1, at this point, the decision rule is

$$\begin{aligned} F(P_1(w), t_1) &= \max\left\{\Omega(P_{t_1}(w), t_1),\right. \\ &\qquad \left. e^{-r(\Delta t)} E_{t_1}\left[F\left(P_{t_2}(w), t_2\right)\right]\right\}. \end{aligned} \quad \text{(B.25)}$$

The optimal stopping time along the w^{th} path satisfies the condition

$$\tau(w) = \inf\{t \mid F(P_t(w), t) = \Omega(P_t(w), t)\} \text{ with } t \in \{t_0, ..., t_N\}. \quad \text{(B.26)}$$

[23] The LSM approach can be applied in non-Markovian settings.

This means that the decision maker will invest the first time the value of the project is superior or equal to the expected continuation value. Finally, to find $F(P_0, t_0)$, we average the discounted path wise values:

$$F(P_0, t_0) = \frac{1}{K} \sum_{w=1}^{K} e^{-r\tau(w)} \Omega\left(P_{\tau(w)}(w), \tau(w)\right). \tag{B.27}$$

At each step along each path, the unknown is the expected continuation value. At node t_n the expected continuation value is equal to

$$\Psi(P_{t_n}, t_n) = e^{-r\Delta t} E_{t_n}\left[F\left(P_{t_{n+1}}, t_{n+1}\right)\right]. \tag{B.28}$$

The key insight underlying the LSM approach is that (B.28) can be expressed as the following combination of basis functions

$$\Psi(P_{t_n}, t_n) = \sum_{j=1}^{\infty} \phi_j(t_n) L_j(P_{t_n}, t_n). \tag{B.29}$$

Where $\phi_j(t_n)$ are the coefficients of the basis functions and L_j is the j^{th} element of the expectation's orthonormal basis.[24] To approximate (B.29) we use the first $J < \infty$ elements of the basis and we estimate the coefficients by least squares with the cross sectional information of the simulation. With this we obtain an approximation of the continuation value.

If we proceed recursively, starting at node $N-1$, the continuation value along each path w is equal to

$$e^{-r\Delta t} \max\left\{\Omega\left(P_{t_N}(w), t_N\right), 0\right\}. \tag{B.30}$$

To obtain an estimate of (B.29) at t_{N-1}, we regress (B.30) for each path on a constant and the J basis functions. Consequently at t_{N-1} for path w, the

[24]For technical details see Longstaff and Schwartz (2001).

continuation value is estimated by

$$\widehat{\Psi}\left(P_{t_{N-1}}(w), t_{N-1}\right) = \sum_{j=1}^{J} \widehat{\phi}_j(t_{N-1}) L_j\left(P_{t_{N-1}}(w), t_{N-1}\right) \tag{B.31}$$

where $\widehat{\phi}_j(t_n)$ is the least square estimate of the j^{th} basis coefficient. Finally, the value of the investment opportunity at node $N-1$ is equal to

$$\begin{aligned} F\left(P_{t_{N-1}}(w), t_{N-1}\right) &= \max\left\{\Omega\left(P_{t_{N-1}}(w), t_{N-1}\right),\right. \\ &\qquad \left.\widehat{\Psi}\left(P_{t_{N-1}}(w), t_{N-1}\right)\right\}. \end{aligned} \tag{B.32}$$

To obtain an approximation of the expected continuation value at node $N-2$, we proceed as previously by using the discounted value of (7) for the continuation value along each path. We repeat this procedure to obtain an estimate of $F(P_0, t_0)$.